PRELIMINARY RESULTS FOR COOPERATIVE EXTENSIONS OF THE BAYESIAN GAME*

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Abstract

The descriptive theory of cooperative game with incomplete information developed to date is surveyed. The theory has the potential to provide game-theoretical foundations of economic analysis of the free societies in which organizations (coalitions) as corporations institute a non-market resource allocation mechanism while using the market resource allocation mechanism at the same time. The present-day corporations are interdependent, so the required game theory needs to model an environment in which the feasibility and implications of coordinated strategy choice within a coalition are influenced by the outsiders' strategy choice.

The first part of the paper provides the key ingredients. After formulating the basic one-shot model, which synthesizes Harsanyi's Bayesian game and Aumann and Peleg's non-side-payment game (NTU game), and illustrating economic examples, two required conditions on an endogenously determined strategy are discussed: measurability with respect to an information structure, and Bayesian incentive compatibility. Several descriptive solution concepts that have been proposed to date are discussed.

The second part addresses six issues studied in the literature: First, the existence of the descriptive solutions in the general setup. Second, explanation of information revelation, that is, a process through which private information turns into public information. Third, definitions of efficiency. Fourth, comparisons of several core concepts. Fifth, the existence results specific only to the Bayesian pure exchange economy, and revival of the core convergence theorem within the framework of the Bayesian pure exchange economy. Sixth, another view on coalition formation, specifically analyses of situations in which coalitional membership is anonymous.

1 Introduction

Since the 1970s, we have seen voluminous literature on the analysis of economic problems with asymmetric information. Harsanyi's (1967/68) Bayesian game and Bayesian equilibrium have served as a conceptual foundation for these analyses. While the literature provided new insights into the workings of the present-day economy which the traditional neoclassical paradigm failed to analyze, many works actually postulated a quite restrictive mode of players' interaction, that is, the principal-agent relationship (Stackelberg's leader-follower relationship), a specific instance of the noncooperative game.

Parallel to the development of Bayesian analyses of the noncooperative game, there has been development in static descriptive cooperative game theory, the theory which analyzes another interactive mode in which several players, with all their diverse (most likely conflicting) interests, come to form a coalition to make a coordinated choice of strategies, because by doing so everybody in the coalition ends up better off than behaving alone (noncooperatively). Aumann and Peleg's (1960) model of non-side-payment game (NTU game) and Scarf's core nonemptiness theorem for this game (see, e.g., Scarf (1973, theorem 8.3.6, p. 211)) serve as a breakthrough in the literature. It was with this model that economists could start analyzing cooperative behavior without imposing problematic conditions on utilities, such as the cardinal nature or the transferability. Scarf's theorem is a milestone in studies of the core, a central descriptive cooperative solution.

Wilson (1978) pioneered the study of cooperative behavior in the pure exchange economy with asymmetric information. During the 1990s, there has been a growing literature on the cooperative game with asymmetric information.

The purpose of this paper is to survey the new strand of cooperative extensions of the Bayesian game. This strand of research is far from complete; on the contrary, there are many unsolved questions, and in fact up to now there has not been any definitive general work – hence the title of our paper. Nevertheless, we decided to take on this task (i.e., to survey the cooperative game theory with asymmetric information), because this game-theoretical area is likely to advance economic theory in a fundamental way. The area is indispensable, for example, in the analysis of an economy with organizations as production units, in particular in the analysis of resource allocation mechanisms instituted in organizations as superior alternatives to the market mechanism. Firms (organizations) in the present-day free societies are interdependent, so we emphasize a general game-theoretical model in which the feasibility and implications of coordinated strategy choice within a coalition are influenced by the outsiders' strategy choice. Another reason for this survey is to clarify the nature of the various approaches proposed to date.

While the conventional noncooperative Bayesian analyses sometimes have assumed the presence of a mediator for the firm activities, there is no need for a mediator in the cooperative Bayesian analysis. Indeed, in reality, corporations are operated without consulting with a mediator; the managers at various levels of corporate hierarchy are not mediators but players in a coalition pursuing their own interests. While the principal-agent theory explains institution of a mechanism as a solution to the mediator's optimization problem, cooperative Bayesian analysis explains it as an endogenously determined strategy bundle chosen by the insiders of the coalition.

The first part of the paper provides the key ingredients, such as the basic one-shot model, the two conditions that an endogenously determined strategy is required to satisfy, and several descriptive solution concepts. The second part reviews several issues that the literature has addressed to date. As is the case with the past major advances in economic theory, the area is expected to develop in fundamentally new directions, as researchers open up further new inciting issues.

2 Basic Ingredients

This section presents the basic model (subsection 2.1), several examples (subsection 2.2), meaningful conditions that endogenous variables (strategies) are required to satisfy (subsections 2.3-2.4), and several descriptive solution concepts (subsection 2.5).

2.1 One-shot model

We will first present a cooperative extension of Harsanyi's (1967/1968) Bayesian game. For full analyses of cooperative behavior, the required model needs to treat at least the strategy concept and coalitional attainability concept explicitly, so that it embodies both the ingredients of the Bayesian game and the ingredients of Aumann and Peleg's (1960) non-side-payment game (NTU game).

Let N be a finite set of *players*. The family of nonempty *coalitions* (nonempty subsets of N) is denoted by $\mathcal{N}$. Each player j has a *choice set* (an *action set*) C^j, a *type set* T^j, and a type-profile dependent *von Neumann-Morgenstern utility function* u^j. At a particular stage of the game player j alone knows which member of the set T^j is truly realized; in this sense the realized member t^j is called j's *private information*. The type set T^j will be assumed to be finite throughout this paper.

For $S \in \mathcal{N}$, define $C^S := \prod_{j\in S} C^j$, $T^S := \prod_{j\in S} T^j$, and write $C := C^N$, $T := T^N$ for simplicity. A member of T is called a *type profile*. Generic elements of sets C^S, T^S, C and T are denoted by c^S $(:= (c^j)_{j\in S})$, t^S, c and t, respectively.

The utility function $u^j: \ C \times T \to \mathbf{R}$ associates j's utility level $u^j(c,t)$ to each choice bundle c and type profile t.

The *ex ante* period is defined as the period in which each player does not have his private information, but has an *ex ante* probability on the type profiles T, subjective or objective. An *interim* period (or rather, an *in mediis* period) is defined as a period in which each player already has his private information but does not know the true type profile. Sometimes *the interim* period is more specifically defined as the period in which each player has only his private information and nothing more; in the following, whenever confusion is likely, we will clarify the meaning of the term "*interim*". The *ex post* period is defined as the period in which everybody knows the true type profile.

Given private information t^j, player j holds his subject probability $\pi^j(\cdot \mid t^j)$ on the others' type profiles $T^{N\setminus\{j\}}$. Sometimes we assume that it is derived from an *ex ante* probability π^j on T by the Bayes rule, $\pi^j(t^{N\setminus\{j\}} \mid t^j) = \pi^j(t^{N\setminus\{j\}}, t^j)/\pi^j(T^{N\setminus\{j\}} \times \{t^j\})$. Some works have an *ex ante* probability as a given datum, while others have the more general approach in which *interim* probabilities are given data.

DEFINITION 2.1.1 (Harsanyi, 1967/1968) A *Bayesian game* is a list of specified data,

$$\{C^j,\ T^j,\ u^j,\ \{\pi^j(\cdot \mid t^j)\}_{t^j\in T^j}\}_{j\in N}.$$

The set T^S gives rise to the partition of T: $\{\{t^S\} \times T^{N\setminus S} \mid t^S \in T^S\}$.

Denote by $\mathcal{T}^S$ the algebra on T generated by this partition, and set $\mathcal{T}^j := \mathcal{T}^{\{j\}}$. We call $\mathcal{T}^j$ player j's *private information structure*.

For analysis of cooperative behavior, one first needs to distinguish between feasible and non-feasible coalitional choices. When the players' true type profile is given as t, each coalition S has a set of *feasible* joint choices, defined as subset $\mathbf{C}_0^S(t)$ of C^S. Write for simplicity, $\mathbf{C}_0^j(t) := \mathbf{C}_0^{\{j\}}(t)$. Thus the *feasible-choice correspondence* $\mathbf{C}_0^S : T \to C^S$ is defined for every coalition S. Notice the possibilities that $\mathbf{C}_0^S(t) \neq \prod_{j\in S} \mathbf{C}_0^j(t)$, and that $\mathbf{C}_0^S(t) \neq \mathbf{C}_0^S(t')$ if $t \neq t'$.

Complete information is defined as the situation in which there is no informational problem, i.e., each set T^j is a singleton. In this case we suppress the notation t from $u^j(c,t)$ and $\mathbf{C}_0^S(t)$, and simply write $u^j(c)$ and C_0^S. If, furthermore, player j's utility function depends only upon his choice (i.e., $u^j(c) = u^j(c^j)$), then the set

$$\tilde{V}(S) := \{(u_j)_{j\in S} \in \mathbf{R}^S \mid \exists\, c^S \in C_0^S : \forall\, j \in S : u_j \leq u^j(c^j)\}$$

is the set of all utility allocations attainable in coalition S. By coordinated choice, the members of coalition S can realize any utility allocation in $\tilde{V}(S)$. For different coalitions S and T, sets $\tilde{V}(S)$ and $\tilde{V}(T)$ lie in different Euclidean spaces, so it is analytically convenient to introduce the cylinders in the same space $\mathbf{R}^N$ based on $\tilde{V}(S)$ and $\tilde{V}(T)$. Define, therefore,

$$V(S) := \{u \in \mathbf{R}^N \mid (u_j)_{j\in S} \in \tilde{V}(S)\}.$$

DEFINITION 2.1.2 (Aumann and Peleg, 1960) A *non-side-payment game* is a cylinder-valued correspondence from $\mathcal{N}$ to $\mathbf{R}^N$, i.e., a correspondence $V : \mathcal{N} \to \mathbf{R}^N$ such that

$$[u, v \in \mathbf{R}^N,\ \forall\, j \in S : u_j = v_j] \implies [u \in V(S) \text{ iff } v \in V(S)].$$

A strategy of a player specifies his choice contingent upon a type profile. Formally, a *strategy* of player j is a function $x^j : T \to C^j$. A choice $c^j \in C^j$ may be identified with the constant strategy $t \mapsto c^j$. Denote by X^j the set of all logically conceivable strategies of player j,

$$X^j := \{x^j : T \to C^j\}.$$

For coalition S, define $X^S := \prod_{j\in S} X^j$; it is the set of all logically conceivable coordinated strategies. Write $X := X^N$ for simplicity. The model under construction explains which strategy bundle $x^S := (x^j)_{j\in S}$ is agreed upon by the members of coalition S.

We next introduce the feasible-strategy concept. Suppose the grand coalition is entertaining possible strategy bundle $\bar{x} \in X$. If coalition S is to deviate from N in this situation, the members of S have to know which strategy bundles x^S are feasible for them. The *a priori* given set $F^S(\bar{x})$ ($\subset X^S$) describes precisely these feasible strategies. Feasibility is thus formulated by a feasible-strategy correspondence $F^S : X \to X^S$, which associates to each strategy bundle $\bar{x} \in X$ (which may not be feasible) the set $F^S(\bar{x})$ of all feasible strategy bundles available to coalition S. A feasible strategy bundle as a function $x^S : T \to C^S$ cannot depend on the outsiders' types $t^{N\setminus S}$, since the members of S do not know them. So the strategy bundle x^S depends only upon their own types t^S, that is, it is $\mathcal{T}^S$-measurable. Here, the correspondence $F^S : X \to X^S$ is interpreted as the basic feasibility determined by resource constraints and the characteristics of individual players. The $\mathcal{T}^S$-measurability should not be interpreted as an informational restriction; the latter will be discussed later. Indeed, we will see in subsections 2.3 and 3.2 stronger measurability requirements reflecting information available to each member.

One might simply wish to use the constant correspondence F_0^S defined by

$$F_0^S(\bar{x}) := \{x^S \in X^S \mid x^S \text{ is a } \mathcal{T}^S\text{-measurable selection of } \mathbf{C}_0^S\}$$

as the feasible-strategy correspondence. We take another approach, however, in which the correspondence F^S is arbitrarily given provided

$$\forall\ \bar{x} \in X : F^S(\bar{x}) \subset \{x^S \in X^S \mid x^S \text{ is a } \mathcal{T}^S\text{-measurable selection of } \mathbf{C}_0^S\}.$$

This allows us to take into account the possibility that coalition S's feasibility is influenced by the outsiders' strategy choice.[1]

Majority of the works done to date are on *ex ante* endogenous determination of a strategy bundle,[2] so we consider the situation in which each player

[1] There is redundancy in definitions of $\mathbf{C}_0^S$ and F^S; once we have the feasible-strategy correspondence concept, we can consistently define a feasible-choice correspondence concept.

[2] One of the major goals of this research area is more ambitious: to explain *interim* endogenous determination of a strategy bundle. Since no general result has been successfully established yet in order to achieve this goal, we chose the title of the present survey paper as it is now.

has an *ex ante* probability on T. For simplicity, we also assume that this is a strictly positive objective probability π.

DEFINITION 2.1.3 (Ichiishi and Idzik, 1996) A *Bayesian society* is a list of specified data

$$\mathcal{S} := \Big(\{C^j, T^j, u^j\}_{j \in N},\ \{\mathbf{C}_0^S, F^S\}_{S \in \mathcal{N}},\ \pi \Big)$$

of: (i) a finite set of players N; (ii) a choice set C^j, a finite set of types T^j, and a von Neumann-Morgenstern utility function $u^j : C \times T \to \mathbf{R}$ for each player j; (ii) a feasible-choice correspondence $\mathbf{C}_0^S : T \to C^S$, and a feasible-strategy correspondence $F^S : X \to X^S$ for each coalition S, such that each element of $F^S(\bar{x})$ is a $\mathcal{T}^S$-measurable selection of $\mathbf{C}_0^S$; and (iii) a strictly positive *ex ante* objective probability π on T.

Wilson (1978) pioneered study of the core of a pure exchange economy with incomplete information. He introduced and carefully discussed several core concepts that would allow for the phenomenon of adverse selection. We recall that adverse selection undermines opportunities for insurance in a market, as Akerlof (1970) demonstrated in his model of the market for lemons. Wilson emphasized the role of revelation of private information. When player j is endowed only with his private information structure, he can distinguish two states (i.e., two type profiles), t and t', iff there exists an event $E \in \mathcal{T}^j$ such that $t \in E$ and $t' \notin E$. Likewise, when coalition S is formed and each member somehow fully reveals his information to his colleagues, any member of S can distinguish two states using the pooled information structure $\mathcal{T}^S$. In general, each member j receives only partial information from his colleagues, so the information structure he can use is an algebra $\mathcal{A}^j$ which is finer than his private information structure but is coarser than the fully pooled information structure.

DEFINITION 2.1.4 (Wilson, 1978) A *communication system* for coalition S is an $\#S$-tuple of algebras $\{\mathcal{A}^j\}_{j \in S}$ on T such that

$$\forall\ j \in S : \mathcal{T}^j \subset \mathcal{A}^j \subset \mathcal{T}^S.$$

It is called *null*, if $\mathcal{A}^j = \mathcal{T}^j$ for every $j \in S$. It is called *full*, if $\mathcal{A}^j = \mathcal{T}^S$ for every $j \in S$.

We have followed Harsanyi (1967/1968) in formulating information structures as algebras on the type profile space T. A type profile is a *state* of the nature. Actually, Wilson (1978) and many subsequent authors took a more general approach in which an arbitrary probability space $(\Omega, \mathcal{T}, \pi)$ is given to describe the possible states of the nature, and an arbitrary subalgebra $\mathcal{T}^j$ of $\mathcal{T}$ is also given to describe player j's private information structure, $j \in N$. Harsanyi's type-profile framework ($\Omega = \prod_{j \in N} T^j$) treats the case of extreme asymmetry of information that private information structures $\mathcal{T}^i$ and $\mathcal{T}^j$ are uncorrelated for different players i and j ($\mathcal{T}^i \cap \mathcal{T}^j = \{\emptyset, T\}$ if $i \neq j$), but Wilson's general approach allows for correlation, that is, $\mathcal{T}^i \cap \mathcal{T}^j$ may contain nonempty proper subsets of Ω. For the expository purpose, however, we adopt the type-profile framework in most parts of this paper.

2.2 Examples

We present several economic examples of the Bayesian society (definition 2.1.3).

EXAMPLE 2.2.1 A *Bayesian pure exchange economy*

$$\mathcal{E}_{pe} := \left(\{\mathbf{R}^l_+, T^j, u^j, e^j\}_{j \in N},\ \pi \right)$$

is an economy with l commodities, where N is a finite set of consumers, and for each consumer j, $\mathbf{R}^l_+$ is his consumption set, T^j is his finite type set, $u^j : \mathbf{R}^l_+ \times T \to \mathbf{R}$ is his type-profile dependent von Neumann-Morgenstern utility function, $e^j : T^j \to \mathbf{R}^l_+$ is his initial endowment vector, which depends only upon t^j, and π is an *ex ante* objective probability on T.

The associated Bayesian society $(\{C^j, T^j, u^j\}_{j \in N},\ \{\mathbf{C}^S_0, F^S\}_{S \in \mathcal{N}},\ \pi)$ is defined as follows. The ingredients, N, T^j and π, are already given in economy $\mathcal{E}_{pe}$. Therefore, we only need to define the choice sets C^j and the feasible-strategy correspondences F^S, so that the definition of C^j enables us to use the utility functions u^j of $\mathcal{E}_{pe}$ as the utility functions of the Bayesian society. Define

$$\begin{aligned} C^j &:= \mathbf{R}^l_+, \\ F^S(\bar{x}) &:= \left\{ x^S : T \to C^S \;\middle|\; \begin{array}{l} x^S \text{ is } \mathcal{T}^S\text{-measurable}, \\ \forall\, t : \sum_{j \in S} x^j(t) \leq \sum_{j \in S} e^j(t) \end{array} \right\}. \end{aligned}$$

Thus, the utility function u^j of the Bayesian society depends only on player j's choice and a type profile. Player j's strategy here is a $\mathcal{T}^S$-measurable demand plan, $x^j : T \to \mathbf{R}^l_+$. Some works (e.g., Hahn and Yannelis (1997), Vohra (1999) and Yazar (2001)) re-formulate the model so that j's strategy is a net trade plan, $z^j : t \mapsto x^j(t) - e^j(t^j)$. Demand plan x^j is $\mathcal{T}^S$-measurable iff net trade plan z^j is $\mathcal{T}^S$-measurable. Choice of demand plan versus net trade plan as a strategy affects some results (see proposition 2.4.3, lemma 3.5.1 and theorem 3.5.3). $\square$

EXAMPLE 2.2.2 A *Bayesian coalition production economy*

$$\mathcal{E}_{cp} := \left(\{\mathbf{R}^l_+, T^j, u^j, e^j\}_{j\in N},\ \pi,\ \{Y^S\}_{S\in\mathcal{N}}\right)$$

is an economy with l commodities, where $(\{\mathbf{R}^l_+, T^j, u^j, e^j\}_{j\in N}, \pi)$ represents the consumption sector $\mathcal{E}_{pe}$, and $\{Y^S\}_{S\in\mathcal{N}}$ represents the production sector: Correspondence $Y^S : T \to \mathbf{R}^l$ associates to each type profile t a production set $Y^S(t)$ $(\subset \mathbf{R}^l)$ for coalition S.

The associated Bayesian society $(\{C^j, T^j, u^j\}_{j\in N},\ \{\mathbf{C}^S_0, F^S\}_{S\in\mathcal{N}},\ \pi)$ is defined as follows. The ingredients, N, T^j and π, are already given in economy $\mathcal{E}_{cp}$. Therefore, we only need to define the choice sets C^j and the feasible-strategy correspondences F^S, so that the definition of C^j enables us to use the utility functions u^j of $\mathcal{E}_{cp}$ as the utility functions of the Bayesian society. Define

$$\begin{aligned} C^j &:= \mathbf{R}^l_+, \\ F^S(\bar{x}) &:= \left\{ x^S : T \to C^S \;\middle|\; \begin{array}{l} x^S \text{ is } \mathcal{T}^S\text{-measurable,} \\ \exists\, y : T \to \mathbf{R}^l,\ \mathcal{T}^S\text{-measurable,} \\ \forall\, t \in T : y(t) \in Y^S(t), \\ \sum_{j\in S} x^j(t) \le y(t) + \sum_{j\in S} e^j(t^j) \end{array} \right\}. \end{aligned}$$

$\square$

EXAMPLE 2.2.3 As a corporation grows, its internal structure evolves. One of the major characteristics of the present-day economy is emergence of *firms in multidivisional form*, (*M-form firms* in short), corporations in which several divisions (called profit centers) are operated semiautonomously; see Chandler (1962) for historical development of the M-form firms.

Each division in an M-form firm is, to a significant extent, an independent decision-maker. As an organization itself, it has its own organizational decision-making. Even if we abstract away the intraorganizational issues of a division, we still need to analyze the interorganizational issues among the divisions: As decision-units of the same corporation, these divisions talk to each other and coordinate their production activities. Total profit will then be distributed to the divisions. (The definition of profit in this example is different from the neoclassical definition of profit, in that it need not reflect the cost of resources, such as capital, that are not under the control of the divisions.)

Radner (1992) (1) formulated an interorganizational issue of the divisions as a static model of profit-center game, (2) viewed the core of the game as the equilibrium outcomes, and (3) studied its properties for several interesting cases. One of the key ingredients in Radner's formulation of an M-form firm is the distinction of *marketed commodities* and *nonmarketed commodities*; while a commodity in the former category has a price established in the market outside the firm, a commodity in the latter category has no price and is used only internally. An *intermediate nonmarketed commodity* is a commodity, not available in the market, which is supplied as an output by a division of the firm and is demanded as an input by another division.[3] A central resource allocation problem in an M-form firm then arises: A nonmarketed commodity produced or initially held by a division (say, division i) is transferred to another division (say, division j), and the two divisions have to come up with a mutually agreeable level of payment that j has to make to i in return for the use of the commodity. This problem, customarily called a *transfer payment problem*, addresses determination of prices according to a nonmarket mechanism.

We present Ichiishi and Radner's (1999) model, which introduces asymmetric information to Radner's (1992) model. It may be considered a particular instance of the Bayesian coalition production economy (example 2.2.2), but due to its focus on an information-revelation process, it has a richer structure. We will present the added structure in subsection 3.2.1. For now, we only mention that the *profit center game with incomplete information* is

[3] A division of an M-form firm, for example, produces computers and sells them in the market. The firm's product, computers, is a marketed commodity. For its production, the division needs as inputs computer chips that are produced in another division of the same firm. This chip, designed only for production of the computer, is useless for any other purpose, in particular outside the firm, so it is an intermediate nonmarketed commodity.

a list of specified data of $\mathcal{D} := (\mathcal{E}_{cp}, p)$ of the coalition production economy

$$\mathcal{E}_{cp} := \Big(\{\mathbf{R}^{k_m+k_n}, T^j,\ \text{profit function}, r^j\}_{j\in N},\ \pi,\ \{Y^j\}_{j\in N}\Big)$$

and a price vector of the market commodities $p \in \mathbf{R}^{k_m}_{+}$, where k_m is the number of the marketed commodities and k_n is the number of the nonmarketed commodities, and for each division j, the input-output space $\mathbf{R}^{k_m+k_n}$ is its choice set, T^j is the set of possible types, $r^j : T \to \mathbf{R}^{k_n}$ is the resource function (initial endowment vector of nonmarketed commodities), π is an *ex ante* objective probability on T, and $Y^j(\cdot)$ is the production set. Each division is risk-neutral. $\square$

2.3 Measurability as feasibility of individual actions

This subsection and next subsection present two economically meaningful conditions that the strategies of a Bayesian society (definition 2.1.3) have to satisfy in the presence of differential information: measurability with respect to the available information structure, and Bayesian incentive compatibility. We first discuss the issue of measurability.

Suppose that the grand coalition is entertaining a strategy bundle $\bar{x}$, but that the members of coalition S are contemplating to defect and to take their own strategy bundle $x^S : T \to C^S$, *ex ante* or *interim*.

Suppose the members of S know that a communication system $\{\mathcal{A}^j\}_{j\in S}$ will be available to them at the time of strategy execution, that is, when each member j will make choice according to his strategy x^j. If t is the true type profile, denoting by $A^j(t)$ the minimal element of $\mathcal{A}^j$ that contains t, player j will know at the time of action that the event $A^j(t)$ has occurred, but he will not know which specific state in the event $A^j(t)$ has actually realized. Since he cannot distinguish the states in $A^j(t)$, he cannot take different actions for any two states in $A^j(t)$. This means that his strategy x^j has to be constant on $A^j(t)$.[4] In other words, his strategy has to be $\mathcal{A}^j$-measurable.

[4] A conference on economic theory is held at a respectable university on the outskirts of Istanbul, Turkey, and many theorists from the world over participate in it. One day during the conference, all the American participants have disappeared from the conference site. The rest of the conference participants are told that the Americans are in the old town of Istanbul, enjoying sightseeing. The precise whereabouts of the American group is the Americans' private information and this information has not been made public; they may

Within the general equilibrium framework, Radner (1968) proposed the measurability condition with respect to an available information structure as a feasibility requirement on individual actions. By adopting this feasibility condition for a coalitional framework, we obtain the following condition.

CONDITION 2.3.1 (Radner, 1968) Suppose that the grand coalition is entertaining a strategy bundle $\bar{x}$, but that the members of coalition S are contemplating to defect and to take their own strategy bundle. Suppose the members of S know that a communication system $\{\mathcal{A}^j\}_{j \in S}$ will be available to them at the time of strategy execution. They can take only those strategies $x^S \in F^S(\bar{x})$ such that x^j is $\mathcal{A}^j$-measurable for every $j \in S$.

The *private information case* is defined as the situation in which when the members execute a strategy bundle, member j has only his private information structure $\mathcal{T}^j$, so knows only his true type t^j and the *interim* probability $\pi^j(\cdot \mid t^j)$ on the others' types. In this case the above measurability condition becomes the *private measurability condition* in that each player j's strategy be $\mathcal{T}^j$-measurable. Notice that function $x^j : T \to C^j$ is $\mathcal{T}^j$-measurable iff it is a function only of t^j. We may, therefore, safely write $x^j(t^j)$ (instead of $x^j(t)$) in the private information case. Yannelis (1991) re-emphasized the significance of the measurability condition by introducing the private measurability to the core analysis of the Bayesian pure exchange economy. For the private information case, define the correspondences $F'^S : X \to C^S$ by

$$F'^S(\bar{x}) := \{x^S \in F^S(\bar{x}) \mid \forall\, j \in S : x^j \text{ is } \mathcal{T}^j\text{-measurable.}\}.$$

The *fully pooled information case* is defined as the situation in which when the members execute the strategy bundle x^S, every member j has the pooled information structure $\mathcal{T}^S$. As soon as we go beyond the private information case, it is highly desirable to explain how agents come to pool or

be at Topkapi or at the Blue Mosque. A Japanese participant, not knowing how to enjoy life, stays at the conference; he wants to have coffee or tea during the session break. He cannot make his choice of coffee versus tea contingent upon the American group's location as he does not know it; it is impossible for him, for example, to choose to drink coffee based on the fact that the Americans are visiting Topkapi and to choose tea based on the fact that they are visiting the Blue Mosque. He can plan to drink coffee regardless whether the Americans are at Topkapi or at the Blue Mosque, or can plan to drink tea regardless whether the Americans are at Topkapi or at the Blue Mosque.

share their private information. This issue will be picked up in subsections 2.4 and 3.2. Some authors (see, e.g., Vohra (1999) or Forges, Minelli, and Vohra (2000)) emphasize that the private measurability requirement is too stringent as a feasibility condition. However, we would like to note, as was done by Radner (1968), that the measurability with respect to the information structure of an individual at the time of his action is a fundamental feasibility requirement. Of course, this measurability requirement need not be the private measurability. But as we just remarked, if the measurability requirement is to take account of information revealed by other players, it is highly desirable to model information revelation explicitly as in the rational expectations model (Radner(1979)). Without such an explicit modeling of information revelation, the private measurability requirement seems highly plausible.

2.4 Bayesian incentive compatibility as feasibility of execution of contracts

The next feasibility condition pertains to the feasibility of execution of strategy bundles viewed as "contracts" made within a coalition. In order for a strategy bundle to be agreed upon by the players, each player must feel certain that the strategy bundle will be executed in exact accordance with its terms. This feasibility condition is formulated as the Bayesian incentive compatibility. We start discussions of this condition for the private information case in subsection 2.4.1. We will discuss the same condition, modified for the other cases, in subsections 2.4.2 and 2.4.3.

2.4.1 Private information case

To execute a $\mathcal{T}^j$-measurable strategy $x^j : T^j \to C^j$, player j of type $\bar{t}^j$ needs to take action $x^j(\bar{t}^j)$. We sometimes say throughout the present paper, "player j represents or poses as type $\bar{t}^j$" In the traditional terminology of the mechanism theory, "player j reports his type $\bar{t}^j$." Unlike most of the noncooperative theory, however, *the present cooperative theory does not suppose the existence of a mediator who would receive reports from the players*; in fact no player needs to report his type to any other players or third person who is outside the game. What we mean by this statement is that he simply makes the choice $c^j := x^j(\bar{t}^j) \in C^j$. The other players may observe the action c^j, but does not have to be told about the type $\bar{t}^j$. This interpretation is particularly important when the function x^j is not 1-1 (i.e., when x^j is not fully information-revealing), since j's action c^j does not provide the information to his colleagues as to which of the types $(x^j)^{-1}(c^j)$ is his true type. In the private information case of the one-shot game, player j can make any choice from the range of function x^j and can avoid being caught.

The members of a coalition agree on a strategy bundle, in order to plan a best choice bundle preparing for every contingency. It is essential, therefore, that choices are later made as scheduled. But the private information case in particular creates the incentive for players to misrepresent their true types after a strategy is chosen, which would prevent realization of the planned result, thereby failing to fulfill the purpose of coalition formation. If members of coalition S foresee at the outset that a particular strategy bundle may later induce such misrepresentation, they will not agree to such a bundle. Therefore, the feasible-strategy set is further restricted to those strategies that are Bayesian incentive-compatible in the sense of d'Aspremont and Gérard-Varet (1979). We elaborate on this idea in this section.

Suppose that the grand coalition is entertaining a strategy bundle $\bar{x}$, but that the members of coalition S are contemplating to defect and to take their own strategy bundle $x^S : T \to C^S$ after defection. Let j be any member of S, whose true type is $\bar{t}^j$. If he makes a choice according to the agreement, his conditional expected utility given his true type is

$$
\begin{aligned}
&Eu^j(x^S, \bar{x}^{N\setminus S} \mid \bar{t}^j) \\
&:= \sum_{t'^{N\setminus\{j\}} \in T^{N\setminus\{j\}}} u^j\left(x^j(\bar{t}^j), x^{S\setminus\{j\}}(t'^{S\setminus\{j\}}), x^{N\setminus S}(t'^{N\setminus S}), (\bar{t}^j, t'^{N\setminus\{j\}})\right) \pi(t'^{N\setminus\{j\}} \mid \bar{t}^j).
\end{aligned}
$$

If on the other hand he makes choice $c^j \in x^j(T^j) \setminus \{x^j(\bar{t}^j)\}$ contrary to the agreement,[5] his conditional expected utility given his true type is

$$
\begin{aligned}
&Eu^j(c^j, x^{S\setminus\{j\}}, \bar{x}^{N\setminus S} \mid \bar{t}^j) \\
&:= \sum_{t'^{N\setminus\{j\}} \in T^{N\setminus\{j\}}} u^j\left(c^j, x^{S\setminus\{j\}}(t'^{S\setminus\{j\}}), x^{N\setminus S}(t'^{N\setminus S}), (\bar{t}^j, t'^{N\setminus\{j\}})\right) \pi(t'^{N\setminus\{j\}} \mid \bar{t}^j).
\end{aligned}
$$

His colleagues $S \setminus \{j\}$ cannot catch this betraying act in the private information case, being led to believe that j's true type were in the event $(x^j)^{-1}(c^j)$.

[5] Recall the present emphasis that players take actions (make choices) rather than make reports; in fact they do not have to report their types to anybody.

In the present context, the *Bayesian incentive compatibility* says that strategy bundle x^S is designed so that nobody has the incentive to act contrary to the promised strategy bundle x^S.

CONDITION 2.4.1 (d'Aspremont and Gérard-Varet, 1979) Suppose that the grand coalition is entertaining a strategy bundle $\bar{x}$, but that the members of coalition S are contemplating to defect and to take their own strategy bundle after defection. In the private information case, members of S agree only on those strategies $x^S \in F'^S(\bar{x})$ that are *Bayesian incentive-compatible*, that is,

$$\begin{aligned}&\forall\ j \in S : \forall\ \bar{t}^j \in T^j : \forall\ c^j \in x^j(T^j) : \\ &\qquad Eu^j(x^S, \bar{x}^{N\setminus S} \mid \bar{t}^j) \geq Eu^j(c^j, x^{S\setminus\{j\}}, \bar{x}^{N\setminus S} \mid \bar{t}^j).\end{aligned}$$

In accordance with d'Aspremont and Gérard-Varet's original formulation, Bayesian incentive compatibility can also be stated as follows: After the members of coalition S have agreed on a strategy bundle $x^S \in F^S(\bar{x})$, they play the Bayesian game

$$\begin{aligned}&\mathcal{BG}(x^S, \bar{x}^{N\setminus S}) \\ &\quad := \ \left\{T^j,\ T^j,\ \{Eu^j(x^S(t^S), \bar{x}^{N\setminus S} \mid t^S)\}_{t^S \in T^S},\ \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\right\}_{j \in S}\end{aligned}$$

where the expectation is taken with respect to the outsiders' type profiles $t^{N\setminus S}$ (see definition 2.1.1). Here, player j's choice space is his type space T^j, so that his strategy is a *pretension function* $\sigma^j : T^j \to T^j$, specifying for each possible true type $\bar{t}^j$ his reported type $\sigma^j(\bar{t}^j)$. The members of S are assuming that the outsiders $N \setminus S$ do not pretend but truthfully execute the strategy bundle $\bar{x}^{N\setminus S}$. When the true type profile of S is $\bar{t}^S$, each member j's utility level is the conditional expected utility $Eu^j((x^i(\sigma^i(\cdot)))_{i\in S}, \bar{x}^{N\setminus S}(\cdot) \mid \bar{t}^S))$ given $\bar{t}^S$. Of course, at the time of playing this Bayesian game, each player $j \in S$ has only his private information, $\bar{t}^j$. A *Bayesian equilibrium* is a strategy bundle σ^{*S} such that for any possible true type profile $\bar{t}^S$, the choice bundle $\sigma^{*S}(\bar{t}^S)$ is a Nash equilibrium:

$$\begin{aligned}&\forall\ \bar{t}^S : \forall\ j \in S : \forall\ t^j : \\ &\qquad Eu^j((x^S \circ \sigma^{*S}, \bar{x}^{N\setminus S}) \mid \bar{t}^j) \geq Eu^j((x^j(t^j), x^{S\setminus\{j\}} \circ \sigma^{*S\setminus\{j\}}, \bar{x}^{N\setminus S}) \mid \bar{t}^j),\end{aligned}$$

where $x^j(t^j)$ is the constant function, $t'^j \mapsto x^j(t^j)$. Bayesian incentive compatibility says that the identity function from T^S to T^S is a Bayesian equilibrium. Thus, player j finds it to his advantage to make an honest report assuming that the others $N \setminus \{j\}$ are also making honest reports.

Bayesian incentive compatibility had been used in Myerson's (1984) study of the λ-transfer value in the context of incomplete information. Ichiishi and Idzik (1996) introduced Bayesian incentive compatibility to the Bayesian core analysis (or more generally, to the Bayesian strong equilibrium analysis).[6]

For the private information case, define the correspondences $\hat{F}^S : X \to C^S$ by

$$\hat{F}^S(\bar{x}) := \{x^S \in F'^S(\bar{x}) \mid x^S \text{ is Bayesian incentive-compatible.}\}.$$

In many interesting private information case, members of the set $\hat{F}^S(\bar{x})$ are abundant; any constant function, for example, is a member of $\hat{F}^S(\bar{x})$ provided it is a member of $F^S(\bar{x})$.

A strategy bundle x^S is called *strictly Bayesian incentive-compatible* relative to the grand coalition's strategy bundle $\bar{x}$, if the Bayesian incentive compatibility condition, expressed by the weak inequalities, is satisfied with strict inequalities, i.e.,

$$\begin{aligned}&\forall\ j \in S : (\forall\ \bar{t}^j, \tilde{t}^j \in T^j : \bar{t}^j \neq \tilde{t}^j) : \\ &\qquad Eu^j(x^S, \bar{x}^{N\setminus S} \mid \bar{t}^j) > Eu^j(x^j(\tilde{t}^j), x^{S\setminus\{j\}}, \bar{x}^{N\setminus S} \mid \bar{t}^j).\end{aligned}$$

A sufficient condition for the existence of a strictly Bayesian incentive-compatible strategy bundle can be found in social choice theory:

LEMMA 2.4.2 (Abreu and Matsushima, 1992) *Let $(\{C^j, T^j, u^j\}_{j\in N}, \{\mathbf{C}_0^S, F^S\}_{S\in\mathcal{N}},\ \pi)$ be a Bayesian society. Assume for each player $j \in N$ that his choice set C^j is convex, his utility function u^j is defined on $C^j \times T^j$, and the function $u^j(\cdot, t^j)$ is affinely linear on C^j for every $t^j \in T^j$. Assume also that there exists a finite subset C_f^j of C^j such that*

$$\begin{aligned}&(\forall\ t^j, t'^j \in T^j : t^j \neq t'^j) : \exists\ c^j, c'^j \in C_f^j : \\ &\qquad u^j(c^j, t^j) > u^j(c'^j, t^j) \quad \textit{and} \quad u^j(c'^j, t'^j) > u^j(c^j, t'^j).\end{aligned}$$

[6] The first draft of Ichiishi and Idzik (1996) had been circulated since the summer of 1991.

Then, there exists a 1-1 function $x^j : T^j \to C^j$ such that

$$(\forall\ \bar{t}^j, \tilde{t}^j \in T^j : \bar{t}^j \neq \tilde{t}^j) : u^j(x^j(\bar{t}^j), \bar{t}^j) > u^j(x^j(\tilde{t}^j), \bar{t}^j).$$

The dependence of j's utility function u^j only on j's own choice and type in lemma 2.4.2 may be called the *no-externality case*, but the other players still influence j through the feasible-strategy correspondences, F^S, $S \ni j$. For application of the above lemma to strict Bayesian incentive compatibility, it suffices to notice that $Eu^j(x \mid t^j) = u^j(x^j(t^j), t^j)$ in the present no-externality case.

Hahn and Yannelis (1997) noted that for the Bayesian pure exchange economy in the private information case, the private measurability condition (condition 2.3.1 for the null communication system) implies the Bayesian incentive compatibility condition (condition 2.4.1), provided that the consumers' strategies are net trade plans:

PROPOSITION 2.4.3 (Hahn and Yannelis, 1997) *Let $\mathcal{E}_{pe}$ be the Bayesian pure exchange economy in the private information case, in which each player j's strategy is his net trade plan and the coalitional feasibility is defined by the equality of supply and demand within the each coalition. Then the measurability condition implies the Bayesian incentive compatibility condition.*

Proof Let $z^S : T^S \to \mathbf{R}^{l \cdot \# S}$ be a feasible strategy bundle of coalition S which satisfies the measurability condition. Then, each z^j is a function of t^j, and

$$\forall\ j \in S : \forall\ t^S : z^j(t^j) = -\sum_{i \in S \setminus \{j\}} z^i(t^i).$$

Let $\bar{t}^S$ be the true type profile, and choose any consumer $j \in S$ and any of his types $\tilde{t}^j$. The attainability has to be satisfied for the two type profiles $(\tilde{t}^j, \bar{t}^{S \setminus \{j\}})$ and $\bar{t}^S$, so

$$z^j(\tilde{t}^j) = -\sum_{i \in S \setminus \{j\}} z^i(\bar{t}^i) = z^j(\bar{t}^j).$$

Then,

$$\begin{aligned} Eu^j(z^j(\tilde{t}^j) + e^j \mid \bar{t}^j) &= Eu^j(z^j(\tilde{t}^j) + e^j(\bar{t}^j) \mid \bar{t}^j) \\ &= Eu^j(z^j(\bar{t}^j) + e^j(\bar{t}^j) \mid \bar{t}^j) \\ &= Eu^j(z^j + e^j \mid \bar{t}^j), \end{aligned}$$

so strategy z^j is Bayesian incentive-compatible. $\square$

Notice that the feasibility of strategies in this proposition is given by the equality (of the total demand and the total supply), but the feasibility in example 2.2.1 is given by the weak inequality. It turns out that under the weak monotonicity assumption on the preference relations, if some measurable strategy bundle satisfies the market clearance condition with weak inequality, then there is a larger measurable strategy bundle which satisfies the market clearance condition with equality (see lemma 3.2.6).

This proposition is no longer valid if a demand plan is used as a strategy. Consider, for example, the following economy with one commodity ($l = 1$): Consumer j's type space consists of two elements, $T^j = \{a^j, b^j\}$, his utility function is the identity function, $u^j(c^j, t) = c^j$, and his initial endowment is given by

$$e^j(t^j) = \begin{cases} 1, & \text{if } t^j = a^j, \\ 2, & \text{if } t^j = b^j. \end{cases}$$

Then the initial endowment bundle e is attainable in N with equality and satisfies measurability. But

$$Eu^j(e^j(b^j) \mid a^j) = 2 > 1 = Eu^j(e^j \mid a^j),$$

so strategy e^j is not Bayesian incentive-compatible. The proposition is not valid either in the general model of Bayesian society $\mathcal{S}$

REMARK 2.4.4 Recall the revelation principle, established for the principal-agent theory of mechanism design: There is one principal (the Stackelberg leader), and a finite set N of agents who have private information (the Stackelberg followers). Let M^j be agent j's message space, and set $M := \prod_{j \in N} M^j$. Let Z be an outcome space. First, the principal designs a mechanism $g : M \to Z$ which specifies an outcome to each message profile, and offers it to the agents. Each agent then decides to accept or reject it. If all agents accept it, each agent j sends a message to the principal; he can condition his message on his private information, so his strategy is a function, $\sigma^j : T^j \to M^j$, which means that agent j sends message $\sigma^j(t^j)$ if his true type is t^j. Denoting by $u^j : Z \times T^j \to \mathbf{R}$ j's type-dependent von Neumann-Morgenstern utility function defined on the outcome space, the agents play the Bayesian game $\{M^j, T^j, u^j \circ g, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\}_{j \in N}$. A strategy bundle

σ gives rise to agent j's *interim* conditional expected utility $Eu^j(g(\sigma) \mid \bar{t}^j)$ given his type $\bar{t}^j$:

$$Eu^j(g(\sigma) \mid \bar{t}^j) := \sum_{t \in T} \pi^j(\{t\} \mid \bar{t}^j) u^j(g(\sigma(t)), \bar{t}^j).$$

A *Bayesian equilibrium* of this game is a strategy bundle σ^* such that for all player $j \in N$ and all his possible type $t^j \in T^j$, his strategy σ^{*j} gives the best response (choice of a message) to the others' strategies $\sigma^{*N\setminus\{j\}}$, that is,

$$\forall\ m^j \in M^j : Eu^j(g(\sigma^*) \mid t^j) \geq Eu^j(g(m^j, \sigma^{*N\setminus\{j\}}) \mid t^j).$$

Now, the *revelation principle* guarantees that, in designing an optimal mechanism, the principal can restrict the set of admissible mechanisms only to those mechanisms from T to Z (that is, each agent j's message space is his type space T^j) that satisfy Bayesian incentive compatibility. To be precise, it says: *Let $g : M \to Z$ be a mechanism, and let σ^* be a Bayesian equilibrium of g. Set $z^*_g := g \circ \sigma^*$, the equilibrium random outcome. Then, there exists a mechanism $g' : T \to Z$ such that the honest-strategy bundle (the identity function from T to T) is a Bayesian equilibrium of g' and gives rise to z^*_g as its equilibrium random outcome.*

The essential implication of the revelation principle is the computational convenience for the principal's designing: The principal may assume *without loss of generality* that his available mechanisms are Bayesian incentive-compatible. In the light of this principle, one might conjecture that also in the Bayesian society the players may choose Bayesian incentive-compatible strategies *without loss of generality*. But this is false; by restricting the feasible-strategy set from $F'^S(\bar{x})$ to $\hat{F}^S(\bar{x})$, coalition S loses substantially, that is, the set of attainable expected utility allocations for S shrinks. By imposing condition 2.4.1, therefore, we are assuming that coalition S pays this loss in order to guarantee truthful execution of an agreed-upon strategy bundle.

To see that the revelation principle does not work here, recall the proof of the revelation principle in the mechanism design theory. It is extremely simple: Let σ^* be a Bayesian equilibrium relative to a given mechanism $g : M \to Z$. Then, the mechanism $g \circ \sigma^* : T \to Z$ is the required Bayesian incentive-compatible mechanism.

For the present Bayesian game $\mathcal{BG}(x^S, \bar{x}^{N\setminus S})$ that follows agreement of a strategy bundle $x^S \in F'^S(\bar{x})$, let σ^{*S} be its Bayesian equilibrium. The problem is that the function $x^S \circ \sigma^{*S} : T \to C^S$, while satisfying the measurability and the Bayesian incentive compatibility conditions, may not be a member of $F^S(\bar{x})$.

As a counterexample to disprove the revelation principle in the Bayesian society, consider the example of the Bayesian pure exchange economy with one commodity given in the paragraph that immediately precedes the present remark. The initial endowment bundle e is an attainable and private measurable strategy bundle. The Bayesian game $\mathcal{BG}(e)$ that follows agreement of e has the unique Bayesian equilibrium σ^*,

$$\sigma^{*j}(a^j) = \sigma^{*j}(b^j) = b^j.$$

The strategy bundle $e \circ \sigma^*$ is a constant function, $e^j \circ \sigma^* : t^j \mapsto 2$, $j \in N$, so is private measurable and Bayesian incentive compatible, but is not attainable. □

So far, we have seen formulations of Bayesian incentive compatibility in the private information case within the framework of one-shot model. More generally, Bayesian incentive compatibility reflects the information system available at the time of action (strategy execution). Subsection 3.2.1 and Appendix present how the definition is modified for a particular two-*interim*-period model.

2.4.2 Mediator-based approach

Vohra (1999) proposed the *mediator-based approach* within the framework of the Bayesian pure exchange economy $\mathcal{E}_{pe}$. There are two possible scenarios for his approach; we first present the one described by Vohra (1999, page 124, the second paragraph). He postulates that there is an enforcement agency (mediator) for each coalition S, who enforces a coalitionally agreed upon strategy bundle, a net trade bundle $\{z^j\}_{j \in S}$, $z^j : T^S \to \mathbf{R}^l$, such that $\sum_{j \in S} z^j(t^S) = \mathbf{0}$ for every $t^S \in T^S$. Notice the dependency of z^j on T^S. When players' types are still private information, each player j communicates his type t^j to the mediator, who, having received the reported type profile t^S, enforces each player j to take the promised action $z^j(t^S)$. The mediator does not know the true type profile, and this fact could create the possibility of j's misrepresenting his type. Suppose $\bar{t}^S$ is the true type profile. Assuming that the others report their true types, j's *interim* expected utility of honest report is

$$\begin{aligned} &Eu^j(z^j + e^j \mid \bar{t}^j) \\ &:= \sum_{t^{N\setminus\{j\}}} u^j\left(z^j(\bar{t}^j, t^{N\setminus\{j\}}) + e^j(\bar{t}^j), (\bar{t}^j, t^{N\setminus\{j\}})\right)\pi(t^{N\setminus\{j\}} \mid \bar{t}^j). \end{aligned}$$

On the other hand, assuming also that the others report their true types, j's *interim* expected utility of dishonest report $\tilde{t}^j$ is

$$\begin{aligned} &Eu^j(z^j(\tilde{t}^j, \cdot) + e^j \mid \bar{t}^j) \\ &:= \sum_{t^{N\setminus\{j\}}} u^j\left(z^j(\tilde{t}^j, t^{N\setminus\{j\}}) + e^j(\bar{t}^j), (\bar{t}^j, t^{N\setminus\{j\}})\right)\pi(t^{N\setminus\{j\}} \mid \bar{t}^j). \end{aligned}$$

Vohra postulates Bayesian incentive compatibility that misrepresentation is not worthwhile. Extending his condition to the Bayesian society is straightforward.

CONDITION 2.4.5 (Vorha, 1999) Coalition S's $\mathcal{T}^S$-measurable strategy bundle z^S in the Bayesian pure exchange economy is Bayesian incentive-compatible, in the sense that

$$\begin{aligned} &\neg\, \exists\, j \in S : \exists\, \bar{t}^j \in T^j : \exists\, \tilde{t}^j \in T^j : \\ &\quad Eu^j(z^j(\tilde{t}^j, \cdot) + e^j \mid \bar{t}^j) > Eu^j(z^j + e^j \mid \bar{t}^j). \end{aligned}$$

Although Vohra emphasizes importance of the mediator's role, in our view the need for a mediator is a weakness of the model in descriptive cooperative theory. We provide, therefore, the second scenario for Vohra's mediator-based approach now; it attempts to eliminate the mediator from the scene. We will conclude, however, that this attempt is not completely accomplished, so that the difficulty remains. *Stage 1*. The members of coalition S agree on a strategy bundle $\{z^j\}_{j\in S}$, assuming that their private information will have been fully pooled by the time of strategy execution, that is, they design each z^j so that it is $\mathcal{T}^S$-measurable. Every member j knows the functional form of his colleague i's strategy z^i. *Stage 2*. At the time the members' true types $\bar{t}^j$, $j \in S$, are private information, they simultaneously and independently communicate their types each other. Assuming honest communication of his colleagues, member j's best action is honest communication of his true type. The true type profile $\bar{t}^S$ is thus transmitted to every member. *Stage 3*. Each member j takes the promised action $z^j(\bar{t}^S)$.

There remains one uneasiness about the above scenario: In stage 2, player j evaluates his possible communication based upon the *interim* probability (the conditional probability given $\bar{t}^j$). At this time, he has not made his choice of net trade yet. In stage 3, however, when he is about to make his choice, he can evaluate his choice based upon the *ex post* probability (the conditional probability given $\bar{t}^S$),[7] and according to this updated probability his decision in stage 2 may not have been optimal. In case his decision in stage 2 turns out to be suboptimal, he may refuse to act as promised in stage 3. This point is illustrated in the following simplest example.

EXAMPLE 2.4.6 Consider the Bayesian pure exchange economy with one commodity ($l = 1$) and two consumers ($N = \{1, 2\}$), in which each consumer's type space has two elements ($T^j = \{t_1^j, t_2^j\}$), all type profiles have the equal *ex ante* probability ($\pi(t) = 1/4$), his utility function depends linearly only on his consumption ($u^j(c^j, t) = c^j$), and his endowment is constant ($e^j(t_1^j) = e^j(t_2^j) = 1$). Define the net trade plan bundle z for the grand coalition by

$$\begin{aligned} z^1(t) &:= \begin{cases} -1 & \text{if} \quad t = (t_1^1, t_1^2) \\ 1 & \text{if} \quad t = (t_1^1, t_2^2) \\ 1 & \text{if} \quad t = (t_2^1, t_1^2) \\ -1 & \text{if} \quad t = (t_2^1, t_2^2), \end{cases} \\ z^2(t) &:= -z^1(t). \end{aligned}$$

This plan satisfies attainability and Bayesian incentive compatibility (condition 2.4.5). Let $\bar{t}$ be any true type profile, say $\bar{t} = (t_1^1, t_1^2)$. Then consumer 1 ends up with the final consumption of 0 at stage 3, which is less than his initial endowment at this state. Consumer 1 will break off from the grand coalition, taking back his initial endowment. $\square$

The mediator-based approach without a mediator thus postulates a corporate (coalitional) atmosphere which forces its members to always act according to an agreed upon strategy bundle. It is this invisible enforcement atmosphere that we label as the "mediator." In reality, however, the effectiveness of this kind of mediator is questionable.

[7] This is the *ex post* probability, indeed, since his net trade plan z^j is $\mathcal{T}^S$-measurable.

The private information case, together with the associated private measurability condition, postulates the safe attitude of each coalition that it avoids to design those mechanisms that could make its member reluctant to act at the time of contract execution.

While the Bayesian incentive compatibility for the private information case (presented in subsection 2.4.1) or for the information-revelation case (via contract execution – to be presented in subsection 3.2.1) reflects the information system available at the time of making a choice (action), the Bayesian incentive compatibility for the mediator-based cases (presented in subsections 2.4.2, 2.4.3 and 3.2.3) does not.

The Bayesian strategic cooperative game theory, pioneered by Wilson (1978), and subsequently developed by Yannelis (1991) as he introduced private measurability, and by Ichiishi and Idzik (1996) as they introduced the general framework and Bayesian incentive compatibility, does not rely on the existence of a mediator. It serves as a theoretical foundation of analyses of the present-day economy, since no mediator, visible or invisible, plays any role in operating the organizations (corporations) in real life. The recent mediator-based approaches left open the question of how to eliminate a mediator.

2.4.3 Communication plan as a part of a strategy

Bayesian incentive compatibility takes quite a different form when players' strategies involve more than type-profile dependent choices. In an attempt to determine Wilson's communication system (definition 2.1.4) endogenously in an equilibrium (core strategy) of the Bayesian pure exchange economy (example 2.2.1), Yazar (2001) defined player j's strategy as a pair of a net trade plan $z^j : T \to \mathbf{R}^l$ and an information substructure $\mathcal{C}^j$. An algebra $\mathcal{C}^j$ on T is called j's *communication plan*, if it is coarser than his private information structure $\mathcal{T}^j$; it is an information structure that j provides to his colleagues as a part of his strategy. When every player i in coalition S chooses strategy $(z^i, \mathcal{C}^i)$, the communication system $\{\mathcal{A}^j\}_{j \in S}$ for S is determined as $\mathcal{A}^j := \mathcal{T}^j \vee (\vee_{i \in S} \mathcal{C}^i)$. It will turn out that this new definition of strategy $(z^j, \mathcal{C}^j)$ is implicit in the blocking concept used in defining Wilson's fine core (definition 2.5.2).

Yazar's scenario goes as follows. Suppose that coalition S is formed and the members choose a strategy bundle $\{z^j, \mathcal{C}^j\}_{j \in S}$. This means in particular that the members of the coalition communicate the information conveyed by $\mathcal{C}^j$ among themselves. Thus, everybody in the coalition has at least the pooled information structure $\vee_{i\in S}\mathcal{C}^i$, so each net trade plan z^j can be made $\vee_{i\in S}\mathcal{C}^i$-measurable.

Let $\bar{t} := \{\bar{t}^j\}_{j\in S}$ be its true type profile. At the beginning of the *interim* period, everybody has only his private information structure, so member j knows that the event $E := \{\bar{t}^j\} \times T^{N\setminus\{j\}}$ has realized; at this moment, any state $t \in E$ could have occurred from j's point of view. Then, everybody passes on information to his colleagues according to the promised communication plan. Let $C^i(t^i) \times T^{N\setminus\{i\}}$ be the minimal element of $\mathcal{C}^i$ that contains t. Member j thinks that if $t \in E$ occurs and if everybody sends the true information, then everybody receives the additional pooled information that the event $\prod_{i\in S} C^i(t^i)$ has realized. Notice that in the light of the measurability requirement, function z^j is constant on $\prod_{i\in S} C^i(t^i)$, so he can choose net trade $z^j(t)$ no matter which state in $\prod_{i\in S} C^i(t^i)$ is true. Consumer j's *interim* expected utility will then be given as

$$\begin{aligned}&Eu^j(z^j(t) + e^j(\bar{t}^j) \mid \bar{t}^j)\\ &\quad := \sum_{t'^{N\setminus\{j\}}\in T^{N\setminus\{j\}}} u^j\left(z^j(t) + e^j(\bar{t}^j), (\bar{t}^j, t'^{N\setminus\{j\}})\right)\pi(t'^{N\setminus\{j\}} \mid \bar{t}^j).\end{aligned}$$

Member j can pass on to his colleagues false information $C'^j \times T^{N\setminus\{j\}} \in \mathcal{C}^j$. Then, he thinks that if $t \in E$ occurs and if everybody else passes on to the others the true information according to the promised communication plan, the additional pooled information is that the event $E' := C'^j \times \prod_{i\in S\setminus\{j\}} C^i(t^i)$ has realized. Function $z^j(t)$ is constant on E'. Member j's *interim* expected utility will be $Eu^j(z^j(E') + e^j(\bar{t}^j) \mid \bar{t}^j)$.

Yazar's condition of Bayesian incentive compatibility says that no member of a coalition can benefit from providing false information to the other members. Extending her condition to the Bayesian society is straightforward.

CONDITION 2.4.7 (Yazar, 2001) Coalition S's strategy bundle $\{z^j, \mathcal{C}^j\}_{j\in S}$ in the Bayesian pure exchange economy is Bayesian incentive-compatible, in the sense that

$$\begin{aligned}&\neg\, \exists\, j \in S : \exists\, \bar{t}^j \in T^j : \exists\, C' \in \mathcal{C}^j : \forall\, t \in \{\bar{t}^j\} \times T^{N\setminus\{j\}} :\\ &\qquad Eu^j(z^j(E') + e^j(\bar{t}^j) \mid \bar{t}^j) > Eu^j(z^j(t) + e^j(\bar{t}^j) \mid \bar{t}^j).\end{aligned}$$

where $E' := C' \times \prod_{i\in S\setminus\{j\}} C^i(t^i)$.

We remark that Yazar's model without a mediator has the same difficulty as Vohra's mediator-based approach without a mediator, that is, having collected the others' private information, some players may not want to act according to an agreed upon strategy bundle.

We presented Yazar's Bayesian incentive compatibility condition within Harsanyi's (1967/1968) type-profile framework, that is, for the case of extreme asymmetry of information, $\mathcal{T}^i \cap \mathcal{T}^j = \{\emptyset, T\}$ if $i \neq j$. She actually adopted Wilson's (1978) general approach in which state space Ω and player j's private information structure $\mathcal{T}^j$ (an algebra on Ω), $j \in N$, are arbitrarily given. Player j's communication plan $\mathcal{C}^j$ is a subalgebra of $\mathcal{T}^j$. His initial endowment e^j is a $\mathcal{T}^j$-measurable function on Ω. Given such a general state space and private information structures, Yazar's original definition of Bayesian incentive compatibility is a bit more involved.

Going back to the type-profile framework, we will point out two facts and argue that Vohra's (1999) work is a special case of Yazar's (2001). Yazar's Bayesian incentive compatibility condition on a strategy bundle with the full communication plan $\{z^j, \mathcal{T}^j\}_{j\in S}$ becomes:

$$
\begin{aligned}
&\neg\, \exists\, j \in S : \exists\, \bar{t}^j \in T^j : \exists\, \tilde{t}^j \in T^j : \forall\, t^{N\setminus\{j\}} \in T^{N\setminus\{j\}} : \\
&\sum_{t'^{N\setminus\{j\}}} u^j\left(z^j(\tilde{t}^j, t^{N\setminus\{j\}}) + e^j(\bar{t}^j), (\bar{t}^j, t'^{N\setminus\{j\}})\right) \pi(t'^{N\setminus\{j\}} \mid \bar{t}^j) \\
&\quad > \sum_{t'^{N\setminus\{j\}}} u^j\left(z^j(\bar{t}^j, t^{N\setminus\{j\}}) + e^j(\bar{t}^j), (\bar{t}^j, t'^{N\setminus\{j\}})\right) \pi(t'^{N\setminus\{j\}} \mid \bar{t}^j).
\end{aligned}
$$

First, she made explicit the measurability of a strategy bundle (condition 2.3.1) with respect to the communication system that is endogenously determined by a communication plan. According to Vohra's scenario, the mediator provides the reported information t^S to each player j (when he tells j to make choice $z^j(t^S)$). This scenario can be viewed as follows: The players have chosen the full communication plan when deciding on the net trade plan z^j. Thus, we can place Vohra's model in Yazar's framework, by viewing Vohra's definition of strategy as a pair $(z^j, \mathcal{T}^j)$ of a net trade plan and the full communication plan. Yazar's condition 2.4.7 on a strategy bundle with the full communication plan $\{z^j, \mathcal{T}^j\}_{j\in S}$ and Vohra's condition 2.4.5 on a $\mathcal{T}^S$-measurable strategy bundle z^S are then the same in sprit. It is true that there is a difference between the two conditions: According to Yazar's condition, j's *interim* conditional expected utility is computed for *each* possible action he may make, $z^j(\tilde{t}^j, t^{N\setminus\{j\}})$, $t^{N\setminus\{j\}} \in T^{N\setminus\{j\}}$, while according to Vohra's condition, j's *interim* conditional expected utility is obtained by integration with respect to his possible actions; in short, there is separate treatment of $t^{N\setminus\{j\}}$ and $t'^{N\setminus\{j\}}$ in Yazar's condition. However, we view that this difference is minor.

Second, in Yazar's framework an arbitrary communication plan (rather than the full communication plan) is possible as a part of a strategy.

2.5 Descriptive solution concepts

Having presented formal models, we are ready to further specify how players interact within the framework of a given model, and present the associated descriptive solution of the game. From the viewpoint of the principal-agent theory we may say, using the terminology of the principal-agent theory, that most of the literature in the Bayesian cooperative theory to date has dealt with the interactive mode in which each player plays both the role of principal and the role of agent: Players get together to make coordinated strategy choice as principals. After the grand coalition decides on its self-sustaining strategy bundle (descriptive solution of the game), each player execute his agreed strategy as an agent in an *interim* period. The solution is called *ex ante* (*interim*, resp.), if it is agreed upon in the *ex ante* period (in an *interim* period, resp.).

A strategy bundle as a function from the type profile space (message-profile space) to the choice-bundle space may be considered a mechanism. While the principal-agent theory of mechanism design explains a mechanism simply as an optimal solution to the principal's problem, the cooperative theory explains it as an endogenous solution to the game.

2.5.1 *Interim* solution concepts

Wilson (1978) paid attention to availability of communication systems (definition 2.1.4) in defining two notions of core within the framework of Bayesian pure exchange economy (example 2.2.1). Although he did not consider the measurability condition (condition 2.3.1) or the Bayesian incentive compatibility condition (condition 2.4.1 or 2.4.5), and the present paper does not consider these conditions either in re-producing his core concepts, the reader can easily incorporate the two conditions in Wilson's core concepts. The reader can also easily extend Wilson's core concepts to strong equilibrium concepts for the Bayesian society (definition 2.1.3).

Given a Bayesian pure exchange economy $\mathcal{E}_{pe}$, define for each coalition S the set of all strategies "feasible" on event $E \subset T$:

$$F_E^S := \left\{ x^S : E \to \mathbf{R}^{l \cdot \# S} \;\middle|\; \forall\, t \in E : \sum_{j \in S} x^j(t) = \sum_{j \in S} e^j(t) \right\}.$$

The conditional expected utility function of commodity allocation plan (strategy) x^j given an algebra $\mathcal{A}^j$ associates with each state $t \in T$ the conditional expected utility of x^j given the minimal element of $\mathcal{A}^j$ that contains t:

$$\begin{aligned} Eu^j(x^j \mid \mathcal{A}^j)(t) &:= Eu^j(x^j \mid A(t)) \\ &= \sum_{s \in T} u^j(x^j(s), s)\pi^j(s \mid A(t)) \end{aligned}$$

where $A(t)$ is the minimal element of $\mathcal{A}^j$ that contains t, and $\pi^j(\cdot \mid A(t))$ is j's conditional probability on T given event $A(t)$ (so, for example, $Eu^j(x^j \mid \mathcal{T}^j)(t) = Eu^j(x^j \mid t^j)$). The following two solutions are *interim* concepts, and are defined for the general situation in which each consumer has subjective *interim* probabilities $\{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}$. The definitions allow for situations in which these *interim* probabilities may not be derived from *one ex ante* probability via the Bayes rule. The first concept is for the situation in which the members of a coalition can use only the null communication system.

DEFINITION 2.5.1 (Wilson, 1978) A commodity allocation plan x^* is said to be in the *coarse core* of Bayesian pure exchange economy $\mathcal{E}_{pe}$, if
(i) $x^* \in F_T^N$; and
(ii) if it is not true that

$$\begin{aligned} &\exists\, S \in \mathcal{N} : \exists\, E \in \wedge_{j \in S} \mathcal{T}^j : \exists\, x^S \in F_E^S : \\ &\quad \forall\, j \in S : \forall\, t \in E : \quad Eu^j(x^j \mid \mathcal{T}^j)(t) > Eu^j(x^{*j} \mid \mathcal{T}^j)(t). \end{aligned}$$

Condition (i) in definition 2.5.1 is feasibility[8] of strategy bundle x^*; the grand coalition is indeed formed in equilibrium and the members jointly

[8] Definitions 2.5.1 and 2.5.2 of coarse core and fine core do not impose the measurability (definition 2.3.1) as a part of the feasibility condition on the solution. It is not clear how far Wilson considered the measurability requirement. In the general formulation of an

choose this bundle. Condition (ii) is the coalitional stability condition, sometimes called the *group incentive compatibility*; it makes precise the idea that no coalition can improve upon x^* in the following sense: No coalition S can have an event E that all members of S can discern ($E \in \wedge_{j\in S}\mathcal{T}^j$) and a feasible consumption plan x^S on E, such that every member j is made better off with x^j than with x^{*j} at each state in E according to his own private information.

In the present extreme case of asymmetric information ($\mathcal{T}^i \wedge \mathcal{T}^j = \{\emptyset, T\}$ if $i \neq j$), the coalitional stability condition for the coarse core becomes: the individual rationality condition,

$$\neg\ \exists\ j \in N : \exists\ t^j \in T^j : \exists\ x^j \in F^j_{\{t^j\}\times T^{N\setminus\{j\}}} : Eu^j(x^j \mid t^j) > Eu^j(x^{*j} \mid t^j),$$

and the coalitional stability condition against non-singletons,

$$\begin{aligned} &\neg\ (\exists S : \#S \geq 2) : \exists\ x^S \in F^S_T : \\ &\quad \forall\ j \in S : \forall\ t^j \in T^j : \ \ Eu^j(x^j \mid t^j) > Eu^j(x^{*j} \mid t^j). \end{aligned}$$

Here, the sets $\{t^j\} \times T^{N\setminus\{j\}}$, $t^j \in T^j$, are the minimal events that singleton $\{j\}$ can discern, and the entire space T is the only event that all members in a non-singleton S, $\#S \geq 2$, can discern. The coarse core as an *interim* solution is based on a very conservative attitude towards coalition-formation: Even when player j has the private information $\bar{t}^j$, he wants to make sure before joining a defecting non-singleton coalition S and agreeing on a joint strategy x^j that he is made better off at every type $t^j \in T^j$, including those that he knows have not realized.

If we weaken the individual rationality condition to

$$\begin{aligned} &\neg\ \exists\ j \in N : \exists\ x^j \in F^j_T : \\ &\quad \forall\ t^j \in T^j : \ \ Eu^j(x^j \mid t^j) > Eu^j(x^{*j} \mid t^j), \end{aligned}$$

which the subsequent literature has done, the resulting weak coalitional stability condition for the coarse core is weaker than the coalitional stability condition for the *ex ante* core; see definitions 2.5.5 and 2.5.6.

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arbitrarily given state space Ω (which may not be a type profile space), in which player j's private information structure is given as an algebra $\mathcal{F}^j$ on Ω, Wilson (1978, page 808, lines 7-10) did require measurability of each strategy with respect to some algebra $\mathcal{F}''$. But the algebra $\mathcal{F}''$ is assumed to be finer than $\vee_{j\in N}\mathcal{F}^j$. Then, in the present formulation of the type profile space ($\Omega = \prod_{j\in N} T^j$ and $\mathcal{F}^j = \mathcal{T}^j$), the algebra $\mathcal{F}''$ is necessarily the finest algebra 2^T, so his measurability does not impose any condition on strategies.

The second concept is for the situation in which each coalition S is *a priori* endowed with a family of *feasible* communication systems, $C(S)$.

DEFINITION 2.5.2 (Wilson, 1978) For each coalition S, let $C(S)$ be an *a priori* given family of feasible communication systems which contains the full communication system.[9] A commodity allocation plan x^* is said to be in the *fine core* of Bayesian pure exchange economy $\mathcal{E}_{pe}$ with $C(S)$, $S \in \mathcal{N}$, if
(i) $x^* \in F^N_T$; and
(ii) if it is not true that

$$\begin{aligned} &\exists\ S \in \mathcal{N} : \exists\ \{\mathcal{A}^j\}_{j \in S} \in C(S) : \exists\ E \in \wedge_{j \in S} \mathcal{A}^j : \exists\ x^S \in F^S_E : \\ &\forall\ j \in S : \forall\ t \in E : \quad Eu^j(x^j \mid \mathcal{A}^j)(t) > Eu^j(x^{*j} \mid \mathcal{A}^j)(t). \end{aligned}$$

An important question that Wilson left open to future research is clarification of the process according to which the members of coalition S come to be endowed with communication systems $C(S)$. More generally, the process of updating information structure from $\mathcal{T}^j$ to a finer structure needs to be explained. Depending upon specification of such information-revelation process, the solution to the game as given in definition 2.5.2 may no longer be appropriate. This issue will be picked up in subsection 3.2.

A coalition is formed when its members agree on a type-profile-contingent choices, which they will execute after the formation. Notice that there is asymmetry between the grand coalition's formation and a blocking coalition's formation, as formulated in the coalitional stability condition for the fine core: The condition includes as one of the requirements,

$$\begin{aligned} &\neg\ \exists\ S \in \mathcal{N} : \exists\ t^S \in T^S : \exists\ x^S \in F^S_{\{t^S\} \times T^{N \setminus S}} : \\ &\forall\ j \in S : \quad Eu^j(x^j \mid t^S) > Eu^j(x^{*j} \mid t^S), \end{aligned}$$

since the full communication system is available. On the one hand, the grand coalition needs to prepare contingencies for all type profiles (x^* is defined on the entire space T). On the other hand, a blocking coalition S needs only to decide on a choice bundle $x^S(t^S) \in C^S$ for its formation. Here, we are

[9] Another interpretation of Wilson (1978, p.813, the third paragraph) is that each family $C(S)$ is given as the family of all communication systems for S, $\{\{\mathcal{A}^j\}_{j \in S} \mid \mathcal{T}^j \subset \mathcal{A}^j \subset \mathcal{T}^S\}$.

assuming that x^S does not depend on the outsiders' types, $t^{N\setminus S}$. So, the domain of a function $x^S \in F^S_{\{t^S\}\times T^{N\setminus S}}$ is the singleton, $\{t^S\}$, and consequently the space $F^S_{\{t^S\}\times T^{N\setminus S}}$ is identified with the choice set $\{c^S \in \mathbf{R}^{l\cdot \# S}_+ \mid \sum_{j\in S} c^j = \sum_{j\in S} e^j(t^j)\}$.

If we postulate that coalition formation requires planning of choices for all contingencies, that is, if we remove the asymmetry problem of the preceding paragraph, and if we consider the private information case, we obtain a related notion of equilibrium which enjoys the strong coalitional stability property that it cannot be improved upon by any coalition regardless of its type profile. In view of the conceptual importance, we present this notion in the most general framework (definition 2.1.3) satisfying the private measurability (condition 2.3.1 for the null communication system) and the Bayesian incentive compatibility (condition 2.4.1):

DEFINITION 2.5.3 Let $\mathcal{S}$ be a Bayesian society, and consider the private information case. A strategy bundle $x^* \in X$ is called an *interim Bayesian incentive-compatible strong equilibrium*, if
(i) $x^* \in \hat{F}^N(x^*)$; and
(ii) it is not true that

$$\begin{aligned} &\exists\ S \in \mathcal{N} : \exists\ t^S \in T^S : \exists\ x^S \in \hat{F}^S(x^*) : \\ &\quad \forall\ j \in S: \quad Eu^j(x^S, x^{*N\setminus S} \mid t^j) > Eu^j(x^* \mid t^j). \end{aligned}$$

The strong equilibrium is an appropriate solution concept for the situations in which coalitions are interdependent, that is, the outsiders' strategy-choice influences the insiders of a coalition through their utility functions or through the coalition's feasible-strategy correspondence. When there is no interdependence, an *interim* Bayesian incentive-compatible strong equilibrium reduces to an *interim Bayesian incentive-compatible core strategy bundle*. The non-interdependence case is obtained when each utility function u^j depends on $C^j \times T$ and each feasible-strategy correspondence F^S is a constant correspondence.

A very specific instance of the *interim* Bayesian incentive-compatible core was used in Ichiishi and Sertel's (1998) study of a profit-center game (example 2.2.3).

In accordance with the mediator-based approach to the Bayesian pure exchange economy $\mathcal{E}_{pe}$, define for $E \in \mathcal{T}^S$

$$F_E^{ic,S} := \left\{ x^S \in F_E^S \;\middle|\; \begin{array}{l} x^S \text{ is } \mathcal{T}^S\text{-measurable, and} \\ \forall\, j \in S : x^j - e^j \text{ is Bayesian incentive-} \\ \text{compatible (condition 2.4.5).} \end{array} \right\}.$$

Vohra (1999) introduced Bayesian incentive compatibility to the coarse core:

DEFINITION 2.5.4 (Vohra, 1999) Given the mediator-based approach, a commodity allocation plan $x^* : T \to \mathbf{R}^{l \cdot \# N}$ is said to be in the *coarse core* of Bayesian pure exchange economy $\mathcal{E}_{pe}$, if
(i) $x^* \in F_T^{ic,N}$; and
(ii) if it is not true that there exist $S \in \mathcal{N}$, $E \in \wedge_{j \in S} \mathcal{T}^j$, and $x^S \in F_E^{ic,S}$ such that

$$\forall\, j \in S : \forall\, t \in E : Eu^j(x^j \mid \mathcal{T}^j)(t) > Eu^j(x^{*j} \mid \mathcal{T}^j)(t).$$

Our final conceptual issue about the *interim* solution is the definition of *contract*. When each player j knows his true type $\bar{t}^j$, it is questionable whether a contract has to specify all his choices contingent on his types that he knows have not realized; that is, a strategy may not be identified with a contract.

In order to see this point, we recall here how the principal-agent theory has addressed this issue. Consider a typical insurance contract theory, in which the insured's probability of causing an accident is his type (his private information), the insurer's type is common knowledge, the insurer is the principal, the insureds are the agents, and the insurer designs a full-coverage insurance policy as a pair $f(t)$ of a premium and a deductible for each possible type t of the insured. Function $f : T \to \mathbf{R}^2$ is a principal's strategy, called a mechanism. The optimal mechanism f^* is the principal's equilibrium strategy. Strategy f^* is not a contract; rather, the image of f^* is interpreted as the set of contracts (insurance policies) he offers to the insureds. Due to Bayesian incentive compatibility, insureds of a specific type t voluntarily choose the contract designed for t, namely $f^*(t)$.

In studying the profit center game with incomplete information (example 2.2.3), Ichiishi and Sertel (1998) interpreted the image $x^*(T)$ of an *interim* core strategy bundle x^* as the set of offered contracts. To appropriately define the *interim* contract concept seems to be determined by specificity of economic contents of the model.

2.5.2 *Ex ante* solution concepts

We turn to the *ex ante* solution concepts. A strategy is identified with a contract here.

Yannelis (1991) addressed the $\mathcal{T}^j$-measurability in the private information case (condition 2.3.1 for the null communication system) in his core analysis of the Bayesian pure exchange economy $\mathcal{E}_{pe}$ (example 2.2.1). We present a somewhat stronger definition than his original private information core concept (see remark 2.5.7 below). The reader can easily extend his core concept to the strong equilibrium concept for the Bayesian society (definition 2.1.3). Recall the definition of $F^S(\bar{x})$ for $\mathcal{E}_{pe}$, which is actually a constant correspondence. Define also

$$F'^S := \left\{ x^S : T \to \mathbf{R}^{l \cdot \# S} \;\middle|\; \begin{array}{l} \forall\, j \in S : x^j \text{ is } \mathcal{T}^j\text{-measurable}, \\ \forall\, t : \sum_{j \in S} x^j(t) \leq \sum_{j \in S} e^j(t) \end{array} \right\}.$$

DEFINITION 2.5.5 (Yannelis, 1991) Let $\mathcal{E}_{pe}$ be a Bayesian pure exchange economy in the private information case. A commodity allocation plan x^* is called a *private information core allocation*, if
(i) $x^* \in F'^N$: and
(ii) it is not true that

$$\exists\, S \in \mathcal{N} : \exists\, x^S \in F'^S : \forall\, j \in S : Eu^j(x^j) > Eu^j(x^{*j}),$$

where $Eu^j(x^j)$ is the ex ante *expected utility of x^j.*

Ichiishi and Idzik (1996) incorporated the Bayesian incentive compatibility condition (condition 2.4.1) in the strong equilibrium analysis:

DEFINITION 2.5.6 (Ichiishi and Idzik, 1996) Let $\mathcal{S}$ be a Bayesian society in the private information case. A strategy bundle $x^* \in X$ is called an *ex ante Bayesian incentive-compatible strong equilibrium*, if
(i) $x^* \in \hat{F}^N(x^*)$; and
(ii) it is not true that

$$\exists\, S \in \mathcal{N} : \exists\, x^S \in \hat{F}^S(x^*) : \forall\, j \in S : Eu^j(x^S, x^{*N \setminus S}) > Eu^j(x^*).$$

The existence question on these *ex ante* solutions will be addressed in subsections 3.1 and 3.5.

The feasibility condition (i) reflects the scenario that the grand coalition is formed in equilibrium. For applications to economies with production, however, we frequently need to explain formation and coexistence of several coalitions (firms). It is easy to extend the Bayesian incentive-compatible strong equilibrium concept, *ex ante* or *interim*, so that a coalition structure is realized in equilibrium (see Ichiishi, 1993a).

REMARK 2.5.7 In both the original definitions of the *ex ante* private core and the *ex ante* Bayesian incentive-compatible strong equilibrium, the inequality in (ii) was replaced by:

$$\forall\, t^j \in T^j :\; Eu^j(x^S, x^{*N\setminus S} \mid t^j) \geq Eu^j(x^* \mid t^j),$$

with strict inequality for at least one t^j. While this formulation clarifies the relationship with Wilson's *interim* coarse core concept or the *interim* coarse strong equilibrium concept (see the second paragraph following definition 2.5.1), the present condition (ii) is a stronger coalitional stability condition. The existence proofs of Yannelis (1991) and Ichiishi and Idzik (1996) actually establish the existence of these stronger solutions. □

Einy, Moreno and Shitovitz (2001b) looked at the bargaining set of a Bayesian pure exchange economy with a nonatomic measure space of consumers.

2.5.3 Other interactive modes

Another interactive mode studied is a multi-principal, multi-agent relationship. The player set N is partitioned into the set of principals and the set of agents. The principals play a cooperative game, taking into account the agents' reactions to their coordinated strategy bundle. While there is no general theory of this mode, Ichiishi and Koray (2000) studied a specific model of education, a version of Spence' model. In their model, the first-stage game played by the principals have the same feature as the prisoner's dilemma game, so there exists no cooperative equilibrium.

In closing this section, we point out a central question left open in the area of Bayesian noncooperative game: how to capture and formulate the *interim* market mechanism in the general equilibrium framework.

The question includes a sensible definition of competitive equilibrium. The required notion is expected to share many features with the rational expectations equilibrium, but we believe that in order to fully describe the individual price-taking behavior in the market, an individual demand function needs to be well defined on the entire price simplex $\Delta^{l-1} := \{p \in \mathbf{R}^l_+ \mid \sum_{h=1}^l p_h = 1\}$, whereas it is defined only on a negligible subset of Δ^{l-1} in the rational expectations equilibrium framework with a finite state space Ω. To see this last point on the rational expectations equilibrium, suppose the government announces message function (price function) $\mathbf{p} : \Omega \to \Delta^{l-1}$. If consumer i observes a (possibly disequilibrium) price vector p, he believes that the event $\mathbf{p}^{-1}(p)$ ($\subset \Omega$) occurs. He then conditions his probability on Ω given $\mathbf{p}^{-1}(p)$, and chooses his demand. But the event $\mathbf{p}^{-1}(p)$ could be empty (or it may have probability 0) unless the observed price p is in the range of function, $\mathbf{p}(\Omega)$. Thus the demand cannot be naturally defined outside the range $\mathbf{p}(\Omega)$. If, for example, Ω is a finite set, then the range $\mathbf{p}(\Omega)$ is also finite, so the demand function is *un*definable almost everywhere.

Once we accomplish the task of formulating the *interim* market mechanism, we can provide a reliable analysis of the market of lemon, and analyses of simultaneous workings of the market resource allocation mechanism and the non-market resource allocation mechanisms instituted in organizations (firms).

3 Issues to Address

We review the issues that have been addressed in the Bayesian cooperative game theory to date: existence theorems for descriptive solutions (subsection 3.1 for the general framework, and subsection 3.5.1 for the Bayesian pure exchange economy); analyses of information revelation processes (subsection 3.2); definitions of Pareto efficiency (subsection 3.3); comparison of the fine core and the *ex post* core, and comparison of implications of the two required conditions, measurability and Bayesian incentive compatibility (subsection 3.4); revival of the core convergence theorem within the framework of the Bayesian pure exchange economy (subsection 3.5.2); and other views on coalition formation, specifically analyses of situations in which coalitional membership is anonymous (subsection 3.6). We also present our view on how to evaluate the existence and the nonexistence results at the end of subsection 3.1.

3.1 Existence

Wilson (1978) established a coarse core nonemptiness theorem, by constructing from the Bayesian pure exchange economy the following non-side-payment game, and by showing that the latter game satisfies the assumptions in Scarf's (1973, theorem 8.3.6, p. 211) core nonemptiness theorem. In Wilson's associated game, a player is defined as a pair (j, t^j) of a consumer and his private information. The admissible coalitions are those of the form,

$$(S, E) := \{(j, t^j) \mid j \in S, t^j \in T^j, \{t^j\} \times T^{N \setminus \{j\}} \subset E\},$$

for $S \in \mathcal{N}$ and $E \in \wedge_{j \in S} \mathcal{T}^j$. The game is defined by

$$V(S, E) := \left\{ u \in \mathbf{R}^{\sum_{j \in N} \# T^j} \;\middle|\; \begin{array}{l} \exists\, x^S \in F_E^S : \\ \forall\, j \in S : (\forall\, t^j : \{t^j\} \times T^{N \setminus \{j\}} \subset E) : \\ u_j \leq E u^j(x^j \mid t^j) \end{array} \right\}.$$

No general existence theorem has been established for the *interim* Bayesian incentive-compatible strong equilibrium or for the *interim* Bayesian incentive-compatible core strategy bundle. So nonemptiness of the fine core also remains open. For a particular numerical example of the Bayesian pure exchange economy with family $C(S)$ which contains both the full communication system and the null communication system, Wilson (1978, p.814) made the following interesting observation in regard to the non-existence of a fine core allocation plan: By considering the blocking behavior of coalitions using the full communication system, the initial endowment is shown to be the only candidate for an unblocked allocation. But the initial endowment is blocked by the grand coalition using its null communication system.

Yannelis (1991) established a private information core allocation existence theorem for an infinite Bayesian pure exchange economy. For the finite economy, an existence theorem can be proved by direct application of Scarf's theorem for nonemptiness of the core (see, e.g., Scarf (1973, theorem 8.3.6, p. 211)) to the non-side-payment game defined by

$$V(S) := \{u \in \mathbf{R}^N \mid \exists\, x^S \in F'^S : \forall\, j \in S : u_j \leq E u^j(x^j)\}.$$

Lefebvre (2001) extended Yannelis' (1991) private information core allocation existence theorem to an infinite Bayesian pure exchange economy with *ex ante* non-ordered preference relations.

REMARK 3.1.1 One can easily extend the coarse core allocation concept and the private information core allocation concept for the Bayesian pure exchange economy to the coarse strong equilibrium concept and the private information strong equilibrium concept for the Bayesian society (definition 2.1.3), and establish existence theorems for these extended concepts. □

Ichiishi and Idzik (1996) established a Bayesian incentive-compatible strong equilibrium existence theorem for the Bayesian society: They first establish an existence theorem for the general class of Bayesian societies with externalities in which each utility function u^j depends fully on $(c, t) \in C \times T$. The assumptions in these theorems are stated, however, partly in terms of derivative concepts such as the parameterized non-side-payment games. Second, they derived from the theorem for the general class an existence theorem for the specific class of Bayesian societies without externalities in which each utility function u^j depends only on $(c^j, t) \in C^j \times T$. Assumptions of the theorem for this specific class are stated only in terms of the exogenously given data $\mathcal{S}$. Notice that in spite of the terminology "no-externalities", the feasible-strategy correspondences F^S depend fully on $x \in X$, and to this extent externalities are still considered. The Scarf theorem for nonemptiness of the core of a balanced non-side-payment game is not applicable here, since a Bayesian incentive-compatible strong equilibrium is a fixed point of the core-strategy correspondence which maps each strategy bundle $\bar{x}$ to the set of core-strategy bundles of the non-side-payment game,

$$V_{\bar{x}}(S) := \left\{ u \in \mathbf{R}^N \;\middle|\; \begin{array}{l} \exists\ x^S \in \hat{F}^S : \forall\ j \in S : \\ Eu^j(x^S, x^{*N \setminus S}) > Eu^j(x^*) \end{array} \right\},$$

but the correspondence is in general disconnected-set valued, so even if we can apply the Scarf theorem to each parameterized game $V_{\bar{x}}$, we cannot obtain a fixed point. Ichiishi and Idzik (1996) applied a social coalitional equilibrium existence theorem (see, e.g., Ichiishi (1993)).

We present the main result of Ichiishi and Idzik (1996) for the no-externality case. Of course, given the no-externality condition, the normal-form games or the Bayesian games are no longer included in the analysis, but economic models extended to cover asymmetric information are included, e.g., the pure exchange economy, the coalition production economy, and the production economy with interdependent organizations that coexist as firms. Due to the full dependence of F^S on $x \in X$, the core of a production economy with public goods can also be analyzed (given the outsiders' production of public goods specified in $x^{N \setminus S}$ the insiders' feasible strategy set $F^S(x)$ describes the sum of these public goods and the insiders' production possibility set).

A subfamily $\mathcal{B}$ of $\mathcal{N}$ is called *balanced* if there exists $\{\lambda_S\}_{S \in \mathcal{B}} \subset \mathbf{R}_+$ such that $\sum_{S \in \mathcal{B}: S \ni j} \lambda_S = 1$ for every $j \in N$.

THEOREM 3.1.2 (Ichiishi and Idzik, 1996) *Let $\mathcal{S}$ be a Bayesian society in the private information case. Assume that each von Neumann-Morgenstern utility function u^j depends only on $C^j \times T$. Assume also*
(i) *for any j, C^j is a nonempty, compact, convex, and metrizable subset of a Hausdorff locally convex topological vector space over $\mathbf{R}$;*
(ii) *for any j and any t, $u^j(\cdot, t)$ is continuous and linear affine in C^j;*
(iii) *for any S and any t, $\mathbf{C}_0^S(t)$ is nonempty, closed and convex;*
(iv) *for any S, correspondence F^S is both upper and lower semicontinuous in X, and has nonempty, closed and convex values;*
(v) *for any $\bar{x} \in X$ and any balanced family $\mathcal{B}$ with the associated balancing coefficients $\{\lambda_S\}_{S \in \mathcal{B}}$, it follows that*

$$\sum_{S \in \mathcal{B}} \lambda_S \tilde{F}^S(\bar{x}) \subset F^N(\bar{x}),$$

where $\tilde{F}^S(\bar{x}) := \{x \mid x^S \in F^S(\bar{x}),\ x^{N \setminus S} = \mathbf{0}\}$;
(vi) *for any S, either F^S is a constant correspondence, or for any S and any $\bar{x} \in X$, there exists $\hat{x}^S \in F^S(\bar{x})$, such that for all $j \in S$ and all $\bar{t}^j, \tilde{t}^j \in T^j$ for which $\bar{t}^j \neq \tilde{t}^j$,*

$$Eu^j(\hat{x}^j \mid \bar{t}^j) > Eu^j(\hat{x}^j(\tilde{t}^j) \mid \bar{t}^j).$$

Then, there exists a Bayesian incentive-compatible strong equilibrium of $\mathcal{S}$.

The affine linearity condition (ii) on $u^j(\cdot, t)$ requires some comments: If choices here are interpreted as *pure choices*, then this assumption imposes the strong condition of risk-neutrality on the players' preference relations. If, on the other hand, choices are interpreted as mixed choices, then the utility here should be interpreted as the expected utility. Of course, the expected utility is linear in probabilities, so the assumption is automatically satisfied under the second interpretation of the choices (that is, the affine linearity does not have to be stated as an assumption); see corollary 3.5.2 below.

Sketch of the Proof of Theorem 3.1.2 We only need to show that the static society $(\{X^j, Eu^j\}_{j\in N},\ \{\hat{F}^S\}_{S\in\mathcal{N}})$ satisfies conditions of a social coalitional equilibrium existence theorem (see, e.g., Ichiishi (1993, Theorem 3.4.11, p. 105)). Upper semicontinuity of each $\hat{F}^S$ is easy to prove. Lower semicontinuity of each $\hat{F}^S$ can be proved by applying the standard argument for lower semicontinuity of the budget-set correspondece in the consumer theory, using assumptions (ii), (iv) and (vi); here the affine linearity of Eu^j on X^j is needed. Using assumptions (ii) and (v), the inclusion of assumption (v) is also satisfied for $\hat{F}^S$; here the affine linearity of Eu^j on X^j is needed again. $\square$

REMARK 3.1.3 The above proof also establishes that the same conditions guarantee the existence of a Bayesian incentive-compatible strong equilibrium of $\mathcal{S}$ for the mediator-based approach (so that each strategy in coalition S is $\mathcal{T}^S$-measurable). $\square$

Ever since Ichiishi and Idzik established an earlier version of theorem 3.1.2 in early summer 1991, the need for the affine linearity assumption on utility functions (ii) when Bayesian incentive compatibility is involved had been known in profession, but in a very specific model (like the pure exchange economy) this assumption can be avoided due to the specific structure of the model and the definition of a strategy (see, e.g., theorem 3.5.3). For an example of a Bayesian society with linear utility functions which does not satisfy the balancedness assumption on feasible-strategy correspondences (v), so has an empty core, see remark 3.5.4.

We present our view on how to evaluate the existence and the nonexistence results. Each economic or game-theoretical model mimics the real world we live in, and each interactive mode specifies players' relationships according to which a game is played. The associated descriptive solution is the outcome that we (analysts) expect to prevail in equilibrium as a result of play of the game, so describes the phenomenon observed in the real world.

If the existence of an equilibrium (e.g., existence of a core allocation or of a Bayesian incentive-compatible strong equilibrium) is guaranteed, we capture the nature of the real phenomena as properties of the equilibrium.

In the case the existence is unlikely, the theorists have held three alternative views on the solution in the past. The first view asserts that the solution is a wrong concept to apply to the real world, so proposes to adopt an alternative solution or even to formulate an alternative model. In the prisoner's dilemma game, for example, a strong equilibrium does not exist. Theorists have sometimes said, "So, the strong equilibrium concept is problematic." Since the unique Nash equilibrium in the same game is not Pareto optimal, theorists have also studied the repeated game (another game) and invoked the folk theorem in order to achieve a Pareto optimal outcome of the original one-shot game.

We contend, however, that the first view does not solve the original problem of understanding the real world; we endorse the following second and the third views. The second view takes the nonexistence result as a warning signal that the model misses important aspects of the reality. To remedy the problem, therefore, we improve the model so that the modified model better reflects the real world and interactive mode.

The third view concerns the situation in which the model captures the essence of the study object, so cannot be improved. Then, we want to analyze a game played within the framework of this model, and not a game in an imaginary world. We also want to study the specific interactive mode that also mimics the pattern of play in the real world. If the associated solution does not exist, we do not apply another solution, since the latter reflects an unrealistic pattern of play. Instead, we accept the fact that the observed phenomena are disequilibrium phenomena, that is, phenomena which we experience in the course of successive formations of blocking coalitions. Thus, if the existence of an equilibrium is not guaranteed, the study object should be the endless formations of coalitions.[10]

3.2 Approaches to information revelation

Each player j is endowed with his private information structure $\mathcal{T}^j$, so he knows his true type $\bar{t}^j$ at the beginning of the *interim* period. By the time the

[10] We quoted the prisoner's dilemma game for the expository purpose. We do not say that this one-shot game mimics the major aspects of the present-day world, or that the associated repeated game is unrealistic. On the contrary, the implications of long-run threat and commitment in the repeated game, as formalized in the folk theorem, constitute a real principle working in the present-day world.

strategy execution is over, player j will have narrowed down the range of his colleague i's possible true types to a subset A_i^j of T^i. In other words, while the players start with the null communication system $\{\mathcal{T}^j\}_{j\in N}$, they end up with an endogenously determined finer communication system $\{\mathcal{A}^j\}_{j\in N}$. This information revelation process is not easy to analyze, since a player j may not want to pass on his private information to his colleagues, and even if j decides to do so, his colleagues may think that j is not truthfully passing on his information but is trying to manipulate them with false information. This subsection will review two approaches taken in the literature for endogenous determination of an information structure: passive information revelation by action; and active information revelation by credible transmission of information (e.g., by credible talking). The first approach is classified into two specific approaches: information revelation by contract execution, and information revelation by choosing a contract. There are works for the information revelation by contract execution and for the information revelation by credible transmission of information; we will review them in subsections 3.2.1 and 3.2.3. We will present in subsection 3.2.2 known examples to illustrate the idea about the information revelation by choosing a contract.

3.2.1 By contract execution

This approach borrows ideas for information update from the rational extpectations equilibrium analysis. In the latter framework, somebody announces a price function $\mathbf{p} : T \to \mathbf{R}^l$ to the economic agents. When an economic agent observes a price vector p, he realizes that the event $\mathbf{p}^{-1}(p) \subset T$ has occurred (see, e.g., Radner (1979)). In the Bayesian society in the private information case, the members of coalition S agree on a strategy bundle x^S, everybody in the coalition knows his collegue i's strategy x^i, so if i makes a choice c^i then the members of S realize that the event $(x^i)^{-1}(c^i) \subset T$ has occurred. Choice is postulated to be observable, so moral hazard problems are excluded.

This subsection presents two works on this information revelation process, Ichiishi, Idzik and Zhao (1994), and Ichiishi and Radner (1999). The essential message of these works is that even if a game starts with the situation characterized as the private information case, it ends up with the full communication system.

Ichiishi, Idzik and Zhao (1994) studied an *ex ante* determination of a strategy bundle, taking into account the above process in the general framework of the Bayesian society $\mathcal{S}$ (definition 2.1.3). For full analysis of players behavior before and after update of information, they introduced an additional structure to $\mathcal{S}$ (postulates 3.2.1 and 3.2.2 below). The first postulate says that each player makes choice twice, once in the first *interim* period, and then in the second *interim* period. The players act simultaneously at each round.

POSTULATE 3.2.1 For each player $j \in N$, his choice set is of the form, $C^j = C_1^j \times C_2^j$.

Set $c^j = (c_1^j, c_2^j)$. Player j makes choice c_1^j (c_2^j, resp.) at his information set of the first *interim* period (the second *interim* period, resp.).

For two information structures $\mathcal{B}$ and $\mathcal{C}$ (algebras on T), denote by $\mathcal{B} \vee \mathcal{C}$ the algebra generated by $\mathcal{B} \cup \mathcal{C}$ (the smallest algebra on T that contains both $\mathcal{B}$ and $\mathcal{C}$). For any set Z and any function $f : T \to Z$, denote by $\mathcal{A}(f)$ the algebra generated by f (the smallest algebra on T that contains the sets $\{f^{-1}(z) \mid z \in Z\}$).

A strategy of player j in coalition S is also denoted by $(x_1^j(\cdot), x_2^j(\cdot))$. Given a strategy bundle x^S, information is processed within coalition S in the following way: In the first *interim* period, each player has only his own private information. So, the component $x_1^j(\cdot)$ has to be $\mathcal{T}^j$-measurable. *If it is common knowledge in S that player j has the incentive to make a choice (say, c_1^j) in the first interim period according to his true type*, then by the beginning of the second *interim* period the occurrence of event

$$\{t^j \in T^j \mid x_1^j(t^j) = c_1^j\}$$

has become common knowledge in S. Let $\bar{t}$ be the true type profile, and suppose choice bundle $c_1^S \in C_1^S$ is made in the first *interim* period. Then each player j has the information that event

$$\{t \in T \mid t^j = \bar{t}^j,\ x_1^S(t) = c_1^S\}$$

has occurred with probability 1.

When designing the other component of the strategy bundle $x_2^S = (x_2^i)_{i \in S}$, the members can anticipate that the information structure,

$$\hat{\mathcal{T}}^i(x_1^S) := \mathcal{T}^i \vee (\vee_{j \in S} \mathcal{A}(x_1^j)),$$

is available to i at the beginning of the second *interim* period, and make each x_2^i measurable with respect to it. Thus, we can make the following postulate of *information-revelation process*:

POSTULATE 3.2.2 Given any strategy bundle $\bar{x} \in X$, coalition S designs only those $x^S \in F^S(\bar{x})$ such that for all $j \in S$ it follows that
(i) x_1^j is measurable with respect to $\mathcal{T}^j$, and
(ii) x_2^j is measurable with respect to $\hat{\mathcal{T}}^j(x_1^S)$.

Denote by $F'^S(\bar{x})$ the set of those feasible strategies x^S that satisfy postulate 3.2.2 (information-revelation process):

$$F'^S(\bar{x}) := \left\{ x^S \in F^S(\bar{x}) \;\middle|\; \begin{array}{l} \forall\, j \in S: \\ x_1^j \text{ is measurable with respect to } \mathcal{T}^j, \text{ and} \\ x_2^j \text{ is measurable with respect to } \hat{\mathcal{T}}^j(x_1^S) \end{array} \right\}.$$

Recall that in order for the present information-revelation process to work, the members of coalition S need to have the common knowledge that each player has the incentive to make a choice in the first *interim* period according to his true type. After all, the contract will not be enforced, if some member has the incentive to make a choice with false pretension about his true type either during the first *interim* period or during the second *interim* period. If the members of S foresee at the time of contract design that a particular contract x^S may later induce such false pretension, they do not agree on the contract x^S. Instead of the strategy set $F'^S(\bar{x})$, therefore, they will consider only the restricted subset $\hat{F}^S(\bar{x})$ of those strategy bundles that are Bayesian incentive-compatible. The definition of Bayesian incentive compatibility in the context of postulates 3.2.1 and 3.2.2 is much involved; see the Appendix for the detail. With the feasible-strategy correspondences $\hat{F}^S : X \to X^S$ modified this way, we can define a *Bayesian incentive-compatible strong equilibrium* as in definition 2.5.6.

The difficulty in establishing an existence theorem for the present Bayesian incentive-compatible strong equilibrium lies in the fact that even if F^S is well-behaved, $\hat{F}^S$ is not convex-valued or upper semicontinuous, so the standard existence techniques do not apply. Ichiishi, Idzik and Zhao (1994) provided generic existence theorems for a Bayesian incentive-compatible strong equilibrium. Again, a generic existence thoerem is established first for the general class of Bayesian societies with externalities in which each utility function

u^j depends fully on $(c,t) \in C \times T$, and then as its application a generic existence theorem is established for the specific class of Bayesian societies without externalities[11] in which each utility function u^j depends only on $(c^j, t^j) \in C^j \times T^j$. Assumptions of the theorem for this specific class are stated only in terms of the exogenously given data $\mathcal{S}$'s; this theorem is presented below (Theorems 3.2.3). Notice again that in spite of the terminology "no-externalities", the feasible-strategy correspondences F^S depend fully on $x \in X$, and to this extent externalities are still considered.

A Bayesian society studied here is a specified list of data,

$$\mathcal{S} := \left(\{C^j, T^j, u^j\}_{j \in N},\ \{C_0^S, F^S\}_{S \in \mathcal{N}},\ \pi\right)$$

(definition 2.1.3), with a rich structure (postulates 3.2.1 and 3.2.2) and the further assumption of no-externalities. One part of the data $(\{C^j, T^j, u^j\}_{j \in N}$, $\{C_0^S\}_{S \in \mathcal{N}},\ \pi)$ will be fixed throughout. By changing the other part of the data from $\{F^S\}_{S \in \mathcal{N}}$ to $\{F^{\dagger S}\}_{S \in \mathcal{N}}$, we obtain *another* Bayesian society

$$\mathcal{S}^\dagger := \left(\{C^j, T^j, u^j\}_{j \in N},\ \{C_0^S, F^{\dagger S}\}_{S \in \mathcal{N}},\ \pi\right).$$

By varying $\{F^S\}_{S \in \mathcal{N}}$, one obtains the *space* of Bayesian societies, $\mathrm{SPACE}_{\mathrm{ne}}$. The space will be endowed with a natural pseudo-metric d. The pseudo-metric space $(\mathrm{SPACE}_{\mathrm{ne}}, d)$ of Bayesian societies will thus be constructed.

A property $\mathcal{P}$ is called a *generic property* of a Bayesian society *in* $\mathrm{SPACE}_{\mathrm{ne}}$, if there exists an open and dense subset $\mathrm{SPACE}'_{\mathrm{ne}}$ of $(\mathrm{SPACE}_{\mathrm{ne}}, d)$ such that every $\mathcal{S} \in \mathrm{SPACE}'_{\mathrm{ne}}$ satisfies $\mathcal{P}$. The following theorem 3.2.3 clarifies conditions on $\mathrm{SPACE}_{\mathrm{ne}}$ under which the following is a generic property of a Bayesian society: *There exist multitude of Bayesian incentive-compatible strong equilibria, and there exists a Bayesian incentive-compatible strong equilibrium x^* such that it fully reveals private information by the end of the first interim period*. Thus, a Bayesian society generically has a Bayesian incentive-compatible strong equilibrium which processes the null communication system to the full communication system through players' actions during the first *interim* period.

In the context of the present structure (postulates 3.2.1 and 3.2.2), the essential role of the feasible-strategy correspondences $\{F^S\}_{S \in \mathcal{N}}$ in Bayesian

[11] The concept of no-externalities is more stringent here than in theorem 3.1.2.

society $\mathcal{S}$ is described by the correspondences $\{G^S\}_{S\in\mathcal{N}}$ defined by

$$G^S(\bar{x}) := \left\{ x^S \in F^S(\bar{x}) \;\middle|\; \begin{array}{l} \forall\, j \in S : \\ x_1^j \text{ is } \mathcal{T}^j\text{-measurable, and} \\ x_2^j \text{ is } \mathcal{T}^S\text{-measurable} \end{array} \right\}.$$

We will give a precise definition of set SPACE$_{\text{ne}}$: It consists of all Bayesian societies satisfying the following conditions 1 through 5. The first two conditions are on the fixed data $(\{C^j, T^j, u^j\}_{j\in N}, \{C_0^S\}_{S\in\mathcal{N}}, \pi)$, hence on the fixed strategy-spaces $X^j := \{x^j : T \to C^j\}$. Conditions 3-5 are on each $\{F^S\}_S$ which defines a member of SPACE$_{\text{ne}}$.

1. (i) For every $j \in N$, his choice set C^j is a nonempty, compact, convex and metrizable subset of a real Hausdorff locally convex topological vector space. (ii) For every $j \in N$, his von Neumann-Morgenstern utility function u^j depends only on $(c^j, t^j) \in C^j \times T^j$, and moreover, $u^j(\cdot\,, t^j)$ is linear affine and continuous on C_0^j for each t^j.

2. For each $j \in N$, there exist $c_m^j \in C^j$ and a finite subset C_f^j of C^j such that (i) for all $c^j \in C_f^j$ and all $t^j \in T^j$, $u^j(c^j, t^j) > u^j(c_m^j, t^j)$; (ii) for all $c^j, c'^j \in C_f^j$ for which $c^j \neq c'^j$, it follows that $c_1^j \neq c_1'^j$; (iii) for all $t^j, t'^j \in T^j$ for which $t^j \neq t'^j$, there exist $c^j, c'^j \in C_f^j$ such that $u^j(c^j, t^j) > u^j(c'^j, t^j)$, and $u^j(c'^j, t'^j) > u^j(c^j, t'^j)$.

3. (i) For each $S \in \mathcal{N}$, correspondence $G^S : X \to C^S$ is upper and lower semicontinuous in X, and for each $\bar{x} \in X$, $G^S(\bar{x})$ is nonempty, closed and convex. (ii) The correspondence $G^N(\cdot)$ is a constant correspondence on X, so one may write $G^N := G^N(\bar{x})$. The set G^N is relatively strictly convex (the strict convex combinations of any two distinct members of G^N are in the relative interior of G^N). There exist $x, x' \in G^N$ such that $Eu(x) \ll Eu(x')$.

4. Choose any $\bar{x} \in X$ and any balanced subfamily $\mathcal{B}$ of $\mathcal{N} \setminus \{N\}$ with the associated balancing coefficients $\{\lambda_S\}_{S\in\mathcal{B}}$. For each $S \in \mathcal{B}$ choose any $(x^{(S)j})_{j\in S} \in G^S(\bar{x})$, and define $\mathcal{T}^j$-measurable strategy $x^j : T^j \to C^j$ by

$$x^j(\bar{t}^j) := \sum_{S\in\mathcal{B}: S\ni j} \lambda_S \left(x_1^{(S)j}(\bar{t}^j), \sum_{t^{S\setminus\{j\}}} \pi(t^{S\setminus\{j\}} \mid \bar{t}^j) x_2^{(S)j}(\bar{t}^j, t^{S\setminus\{j\}}) \right).$$

Then, $x \in G^N$.

5. For each $S \in \mathcal{N}$ one of the following two conditions holds true: (i) $G^S(\cdot)$ is a constant correspondence; or (ii) For each $j \in S$, there exists a finite subset C^j_f of C^j such that for every $\bar{x} \in X$, $\prod_{j \in S}\{x^j : T^j \to \text{ co } C^j_f\} \subset F^S(\bar{x})$, and such that for all $t^j, t'^j \in T^j$ for which $t^j \neq t'^j$, there exist $c^j, c'^j \in C^j_f$ so that $u^j(c^j, t^j) > u^j(c'^j, t^j)$, and $u^j(c'^j, t'^j) > u^j(c^j, t'^j)$. Here, the convex hull of a subset A of a vector space is denoted by co A.

Condition 1 (i) is standard in economic theory. Condition 1 (ii) describes no-externalities, and moreover, imposes a condition which is interpreted in two different ways: First, if each of spaces C^j_1 and C^j_2 consists only of *pure* (or, *deterministic*) *choices*, then it means the risk-aversion. Second, if each of spaces C^j_1 and C^j_2 consists only of *mixed choices* (or *probabilities on pure-choices*) for j, if $u^j(\cdot\ , t^j)$ is interpreted as the expected utility as a function of j's mixed-choice pairs, and if the underlying von Neumann-Morgenstern utility function is additively separable with respect to the pure choice of the first *interim* period and the pure choice of the second *interim* period, then condition (ii) is automatically satisfied.

Conditions 2 and 5 (ii) are made so that they guarantee existence of a *strictly* Bayesian incentive-compatible strategy bundle; see Abreu and Matsushima's lemma (lemma 2.4.2).

Condition 4 is a version of the balancedness condition on the sets $\{G^S(\bar{x})\}_{S \in \mathcal{N}}$, and makes explicit the extent to which the grand coalition has a large feasible-strategy set G^N. It means (1) that for each j the combination of the strategies $\{x^{(S)j}\}_{S \in \mathcal{B}: S \ni j}$ with the convex coefficients $\{\lambda_S\}_{S \in \mathcal{B}: S \ni j}$ is feasible in the grand coalition; and (2) that each member j is insured in the grand coalition to be able to choose $x^j(\bar{t}^j)$ at any state $t \in \{\bar{t}^j\} \times T^{N \setminus \{j\}}$.

Since each C^j is a metric space (Assumption 5.2 (i)) and T is finite, X^j is also a metric space. Denote by ρ_S the Hausdorff distance on the closed subsets of X^S. The pseudo-metric d on SPACE_{ne} is defined by:

$$d(\mathcal{S}, \mathcal{S}^\dagger) := \max_{S \in \mathcal{N}} \max_{\bar{x} \in X} \ \rho_S(G^S(\bar{x}), G^{\dagger S}(\bar{x})).$$

Notice that d may not be a metric, since two distinct sets, $F^S(\bar{x})$ and $F^{\dagger S}(\bar{x})$, may give rise to the identical sets, $G^S(\bar{x}) = G^{\dagger S}(\bar{x})$.

THEOREM 3.2.3 (Ichiishi, Idzik and Zhao, 1994) *Let* $(\text{SPACE}_{\text{ne}},\, d)$ *be the pseudo-metric space of Bayesian societies without externalities satisfying postulates 3.2.1 and 3.2.2. The following is a generic property of a Bayesian society in* $(\text{SPACE}_{\text{ne}},\, d)$*: There exist multitude of Bayesian incentive-compatible strong equilibria, and there exists a Bayesian incentive-compatible strong equilibrium* x^* *such that* x_1^{*j} *is 1-1 on* T^j*.*[12]

Sketch of the proof The proof borrows Radner's (1979) idea for a generic rational expectations equilibrium existence theorem: Given a Bayesian society $\mathcal{S}$, we consider the auxiliary society in which the set $F'^S(\bar{x})$ of feasible strategies satisfying the information-revelation process is replaced by set $G^S(\bar{x})$. The correspondences $\{G^S\}_{S\in\mathcal{N}}$ are well behaved; in particular they are lower semicontinuous and convex-valued, if the originally given feasible-strategy correspondences $\{F^S\}_{S\in\mathcal{N}}$ are lower semicontinuous and convex-valued. Thus, the same technique for proving theorem 3.1.2 is applicable to the auxiliary society. We then prove that the auxiliary societies generically have a Bayesian incentive-compatible strong equilibrium x^* such that the first *interim* period strategy x_1^{*j} is 1-1 for every j. Then x^* is a member of $\hat{F}^N(x^*)$, so is the required equilibrium. □

Ichiishi and Radner (1999) addressed the information revelation process via action within the setup of profit center game with incomplete information $\mathcal{D} := (\mathcal{E}_{cp}, p)$, defined as a pair of a specific Bayesian coalition production economy

$$\mathcal{E}_{cp} := \Big(\{\mathbf{R}^{k_m+k_n}, T^j,\ \text{profit function}, r^j\}_{j\in N},\ \pi,\ \{Y^j\}_{j\in N}\Big)$$

and a price vector of the market commodities $p \in \mathbf{R}_+^{k_m}$ (example 2.2.3). Due to the specific structure of the model, they could establish *exact* existence theorems for a full-information revealing *ex ante* core plan (Bayesian strong equilibrium in the present framework), rather than a mere *generic* existence theorem. Some theorems are valid even for games that are ruled out from space SPACE_{ne}. Of course, postulates 3.2.1 and 3.2.2 are made here. Indeed, the two-*interim*-period framework naturally arises from the economic context: The first *interim* period is the setup period, and the second *interim* period is the manufacturing period. The *setup period* is for the

[12] Function x_1^{*j} is $\mathcal{T}^j$-measurable, iff it may be considered a function only of $t^j \in T^j$. A $\mathcal{T}^j$-measurable function is called 1-1 on T^j, if it is 1-1 as a function defined on T^j.

divisions' simultaneous decisions about initial investment, setting up their manufacturing processes. The *manufacturing period* is for subsequent decisions about actual choice of an input-output vector, and for imputation of the profit that is made by sale/purchase of the market commodities. We will review this work now.

Given a type profile $t \in T$, a *profit imputation* of coalition S is a vector $x^S(t) := (x^j(t))_{j \in S}$ whose jth coordinate is the accounting profit attributed to division j. A *profit imputation plan* of coalition S is a function $x^S : T \to \mathbf{R}^S$, $t \mapsto x^S(t)$. Denoting by $y^S : T \to \mathbf{R}^{(k_m+k_n)\cdot \#S}$ a *net output plan*, a pair (x^S, y^S) will henceforth be called a *plan*.

Let K be the index set of all commodities; it is partitioned into the index set K_m of marketed commodities and the index set K_n of nonmarketed commodities. Let K_1 (K_2, resp.) denote the index set for the commodities that are produced/used in the setup period (in the manufacturing period, resp.). The family $\{K_1, K_2\}$ is a partition of K, possibly different from $\{K_n, K_m\}$. Set $k := \#K$, $k_1 := \#K_1$, $k_2 := \#K_2$. A net output plan y^j may be written as

$$y^j = \begin{pmatrix} y_1^j \\ y_2^j \end{pmatrix},$$

where the components of $y_1^j(t)$ (of $y_2^j(t)$, resp.) correspond to K_1 (K_2, resp.). Define y_m^j and y_n^j similarly corresponding to K_m and K_n. Define also $K_{1n} := K_1 \cap K_n$, $k_{1n} := \#K_{1n}$, and define K_{2n}, K_{1m}, K_{2m}, k_{2n}, k_{1m} and k_{2m} similarly. The initial resource function r^j may be written as

$$r^j = \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix},$$

where r_1^j is a function from T to $\mathbf{R}^{k_{1n}}$, and the components of $r_1^j(t)$ correspond to K_{1n}.

To start the precise description of the scenario, denote by F^S the set of all *technologically attainable* plans of a coalition S, that is, the set of all $\mathcal{T}^S$-measurable functions $(x^S, y^S) : T \to \mathbf{R}^{(1+k)\cdot \#S}$ such that y^S is technologically feasible, i.e.,

$$y^S \in Y^S := \prod_{j \in S} Y^j,$$

and such that the total resource constraint is satisfied within S, i.e.,

$$\forall\ t \in T:\ \sum_{j \in S} \begin{pmatrix} x^j(t) \\ \mathbf{0} \end{pmatrix} \leq \sum_{j \in S} \begin{pmatrix} p \cdot y^j_m(t) \\ y^j_n(t) + r^j(t) \end{pmatrix}.$$

Notice that negative imputation is allowed.[13]

Suppose coalition S is to form. The members can consider only plans which obey the information pooling rule; let F'^S to be the set of those *allowable* plans for S, i.e.,

$$F'^S := \left\{ (x^S, y^S) \in F^S \;\middle|\; \begin{array}{l} \forall\ j \in S: \\ y^j_1 \text{ is } \mathcal{T}^j\text{-measurable}, \\ (x^j, y^j_2) \text{ is } \hat{\mathcal{T}}^j(y^S_1)\text{- measurable} \end{array} \right\}.$$

The members further restrict their plans to those that satisfy the Bayesian incentive compatibility (see Appendix for a precise definition). Let $\hat{F}^S$ be the set of allowable, Bayesian incentive-compatible plans for S.

The Bayesian incentive compatibility may be too stringent a condition that there may not be a strategy in $\hat{F}^N$ which is coalitionally stable. To overcome this difficulty, the headquarters play the role of an insurer. A plan (x^S, y^S) is called *weakly Bayesian incentive-compatible* if for all $j \in S$, and all $\bar{t}^j, \tilde{t}^j \in T^j$, it follows that

$$E(x^j \mid \bar{t}^j) \geq E(x_j \circ (\tilde{t}^j,\ \mathrm{id}) \mid \bar{t}^j).$$

It is not difficult to show that *for a weakly Bayesian incentive-compatible plan (x^S, y^S), the conditional expectation $E(x^j \mid t^j)$ is independent of t^j, that is, $E(x^j \mid \mathcal{T}^j)$ is a constant function*. This fact motivates the following formulation of the postulate of the *headquarters' insurability*:

POSTULATE 3.2.4 Let (x^N, y^N) be a technologically attainable plan of the grand coalition such that it satisfies the information-revelation process, and $E(x^j \mid \mathcal{T}^j)$ is a constant function for each $j \in N$. Then the plan $((E(x^j \mid \mathcal{T}^j))_{j \in N}, y^N)$ is available to the grand coalition N.

[13] In the first main result (theorem 3.2.5), the existence of an equilibrium plan (x^{*N}, y^{*N}), called an *ex ante core plan*, for which $\forall\ t : \forall\ j : x^{*j}(t) \geq 0$ is asserted.

Being an insurer is the only role that the headquarters plays in this game, in addition to participating in the coalitional design of a plan as one of the divisions. By this postulate, division j can receive the accounting profit according to the constant profit imputation plan $E(x^j \mid \mathcal{T}^j)$. This is justified if the headquarters is risk-neutral. Moreover, this is an easy task for the headquarters, because it does not have to know the true type of division j (the need for insurance occurs only when $E(x^j \mid t^j)$ is the same for all t^j). This postulate does not reduce the model to a static game. Indeed, while the profit imputation plan for the grand coalition, $(E(x^j \mid \mathcal{T}^j))_{j \in N}$, stated in the postulate has a static flavor as a constant function, it is made possible by non-constant net output function y^N, and the latter is subject to the information-revelation process.

Let H^N be the set of all plans (x^N, y^N) for the grand coalition N such that x^N is a constant function, and such that there exists $x'^N : T \to \mathbf{R}^{|N|}$ for which $(x'^N, y^N) \in F'^N$ and $E(x'^j \mid \mathcal{T}^j) = x^j$ for every $j \in N$. Notice that every member of H^N is Bayesian incentive-compatible. In the light of the headquarters' insurability, define:

$$\hat{F}^{*S} := \begin{cases} \hat{F}^S, & \text{if } S \neq N, \\ \hat{F}^N \cup H^N, & \text{if } S = N. \end{cases}$$

The set $\hat{F}^{*S}$ is the set of all technologically attainable or insurable plans of coalition S that are consistent with the three postulates, the information-revelation process, the Bayesian incentive compatibility, and the headquarters' insurability. Plan (x^S, y^S) is a candidate for coalition S's agreement, iff $(x^S, y^S) \in \hat{F}^{*S}$.

An *ex ante core plan* of a profit-center game with incomplete information $\mathcal{D}$ is a Bayesian incentive-compatible strong equilibrium: It is a plan (x^{*N}, y^{*N}) of the grand coalition N such that (i) $(x^{*N}, y^{*N}) \in \hat{F}^{*N}$, and (ii) it is not true that there exist $S \in \mathcal{N}$ and $(x^S, y^S) \in \hat{F}^{*S}$ such that $Ex^j > Ex^{*j}$ for all $j \in S$. A core plan (x^{*N}, y^{*N}) is called *full-information revealing*, if for every $j \in N$, y_1^{*j} is 1-1 on T^j. In this case, the updated algebra $\hat{\mathcal{T}}^j(y_1^{*N})$ becomes the full communication system 2^T.

THEOREM 3.2.5 (Ichiishi and Radner, 1999) *Let $\mathcal{D}$ be a profit-center game with incomplete information which satisfies the three postulates: the information-revelation process, the Bayesian incentive compatibility, and the*

headquarters' insurability. Assume for each j,
(i) *the production set Y^j is closed in $\mathbf{R}^{k \cdot \# T}$;*
(ii) $\mathbf{0} \in Y^j$;
(iii) $Y^j - \mathbf{R}_+^{k \cdot \# T} \subset Y^j$;
(iv) *for each $y_n^j \in \mathbf{R}^{k_n \cdot \# T}$, the production possibility set $\{y_m^j \in \mathbf{R}^{k_m \cdot \# T} \mid (y_m^j, y_n^j) \in Y_j\}$ is bounded from above;*
(v) *$K_{1n} \neq \emptyset$, and for each j, the function r_1^j is $\mathcal{T}^j$-measurable, that is, it depends only upon t^j;*
(vi) *the function r_1^j is 1-1 on T^j;*
(vii) *$r^j(t) \geq \mathbf{0}$, for all $t \in T$;*
(viii) *the* ex ante *probability π is the product probability of π^j, $j \in N$, where π^j is a probability on T^j.*
Assume moreover that the production set Y^j is convex for each $j \in N$. Then there exists a full-information revealing ex ante core plan of the game.

Conditions (i)-(iv) in theorem 3.2.5 are the basic assumptions on the production sets. In particular, (ii) says that zero production activity is possible; (iii) means free disposal; and (iv) means the impossibility of the Land of Cockaigne.

Conditions (v)-(vii) are the basic assumptions on the resource function of each division. In particular, (v) says that there are nonmarketed commodities which are used or produced in the first period, and each resource function r_1^j of these commodities depends only on j's type; (vi) says that division j's technology is embodied in its nonmarketed resources of the first period; and (vii) says that all resources are nonnegative.

Before providing a sketch of the proof of theorem 3.2.5, we present a lemma: In this lemma, $y_1^j = (y_{1m}^j, y_{1n}^j)$, so vector $y_{1n}^i(t^i)$ is a nonmarketed commodity bundle used as inputs in the setup period, whose components are measured by negative real numbers according to the usual sign convention on inputs and outputs.

LEMMA 3.2.6 (Ichiishi and Radner, 1999) *Suppose that for each i there exists a function $y_{1n}^i : T^i \to \mathbf{R}^{k_{1n}}$ such that*

$$\forall\ t \in T: \ -\sum_{i \in I} y_{1n}^i(t^i) \leq \sum_{i \in I} r_1^i(t^i).$$

Then, for each i there exists a function $\overset{=}{y}{}^{i}_{1n} : T^i \to \mathbf{R}^{k_{1n}}$ *such that* $\overset{=}{y}{}^{i}_{1n} \le y^i_{1n}$ *and*

$$\forall\ t \in T:\ -\sum_{i \in I} \overset{=}{y}{}^{i}_{1n}\,(t^i) = \sum_{i \in I} r^i_1(t^i).$$

Sketch of the Proof of Theorem 3.2.5 Apply Scarf's theorem for nonemptyness of the core (see, e.g., Scarf (1973, theorem 8.3.6, p. 211)) to the non-side-payment game defined by

$$V(S) := \{u \in \mathbf{R}^N \mid \exists\ (x^S, y^S) \in \hat{F}^{*S} : \forall\ j \in S : u_j \le Ex^j\}.$$

¿From a core plan (x^N, y^N), we obtain another core plan (x^{*N}, y^{*N}) for which total use and total supply of each commodity are equal in view of lemma 3.2.6. Then, for all $t^j, t'^j \in T^j$ and all $t^{N \setminus \{j\}} \in T^{N \setminus \{j\}}$,

$$\begin{aligned} -y^{*j}_{1n}(t^j) - r^j_1(t^j) &= \sum_{i \in N \setminus \{j\}} (y^{*i}_{1n}(t^i) + r^i_1(t^i)) \\ &= -y^{*j}_{1n}(t'^j) - r^j_1(t'^j). \end{aligned}$$

Since r^j_1 is assumed to be 1-1, y^{*j}_{1n} has to be also 1-1. Therefore, plan (x^{*N}, y^{*N}) is full-information revealing. □

Ichiishi and Radner (1999) established further existence results: (1) for the case in which the total production set $\sum_{j \in N} Y^j$ exhibits a specific instance of increasing returns to scale, that is, a stronger version of Scarf's (1986) distributiveness condition; and (2) for the case in which there exists a specific supplier-customer relationship among the divisions.

Ichiishi and Sertel (1998) continued study of the profit-center game with incomplete information. They studied the *interim* Bayesian incentive-compatible core (definition 2.5.3) and welfare loss. Their analysis of the *interim* Bayesian incentive-compatible core is facilitated by the following observation: Due to the weak Bayesian incentive compatibility which is implied by Bayesian incentive compatibility, a strategy bundle $(x^{*N}, y^{*N}) \in \hat{F}^{*N}$ satisfies

$$\forall\ t^j, t'^j \in T^j : E(x^j \mid t^j) = E(x^j \mid t'^j),$$

consequently,

$$\forall\ t^j \in T^j : E(x^j \mid t^j) = Ex^j.$$

Thus, the *interim* core and the *ex ante* core coincide in this model.

3.2.2 By choosing a contract

The next approach to information revelation borrows the idea from the principal-agent theory that agreeing or refusing to sign a contract reveals a private information. It applies to *interim* contracting in the private information case. There is no definitive written work based on this idea, however, and indeed it has a serious limitation if no other approaches are adopted concurrently. We will see two examples first in which a player's intention to sign a contract reveals his private information to the other players.

EXAMPLE 3.2.7 This example, attributed by Ichiishi and Sertel to an anonymous referee of their paper Ichiishi and Sertel (1998), describes a situation in which coalition formation is more difficult than is suggested by the coalitional stability condition of the *interim* Bayesian incentive-compatible strong equilibrium (definition 2.5.3). The essence of this example was observed by Wilson (1978, example 1, p. 809) when he illustrated the phenomenon of adverse selection which often violates opportunities for insurance.

Assume that each choice set is the real numbers, $C^j = \mathbf{R}$, and each utility function is the projection onto C^j, $u^j(c,t) = c^j$. Suppose the grand coalition is deliberating on the constant strategy bundle x:

$$\forall\ j \in N : \forall\ t \in T : x^j(t) = 1.$$

Suppose also that subcoalition $S := \{1,2\}$ finds the following strategy bundle $x'^S \in F^S(x)$: Assume $\pi(t) = \prod_{j \in N} \pi^j(t^j)$. Assume also for each $i \in S$, $T^i = \{H^i, L^i\}$, $\pi^i(H^i) = \pi^i(L^i) = 1/2$.

$$\begin{aligned} x'^1(t) &:= \begin{cases} 4, & \text{if } t = (H^1, L^2), \\ 0, & \text{otherwise;} \end{cases} \\ x'^2(t) &:= \begin{cases} 4, & \text{if } t = (L^1, H^2), \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

Notice that

$$E(x'^i \mid H^i) = 2 > 1 = E(x^i \mid H^i), \text{ for every } i \in S,$$

so S can improve upon x using x'^S when the true type profile is $\bar{t}^S = (H^1, H^2)$. However, player 1 knows that player 2 agrees to the joint strategy x'^S only when 2's true type is H^2, since

$$E(x'^2 \mid L^2) = 0 < 1 = E(x^2 \mid L^2).$$

Then player 2's agreement to x'^S reveals the information to player 1 that 2's true type is H^2. Given this information, player 1 does not agree to x'^S since

$$x'^1(t^1, H^2) = 0 < 1 = x^1(t^1, H^2), \text{ for } t^1 = H^1, L^1.$$

Thus, strategy x'^S cannot serve as a "blocking" strategy against x^S. □

EXAMPLE 3.2.8 This example, a variation of the previous example, describes a situation in which coalition formation is easier than is suggested by the coalitional stability condition of the *interim* Bayesian incentive-compatible strong equilibrium (definition 2.5.3). Assume again $C^j = \mathbf{R}$, $u^j(c,t) = c^j$, $S := \{1,2\}$, $T^i = \{H^i, L^i\}$, $\pi^i(H^i) = \pi^i(L^i) = 1/2$ for each $i \in S$. Suppose the grand coalition is deliberating on the constant strategy bundle x:

$$\forall\, j \in N : \forall\, t \in T : x^j(t) = 1.$$

Suppose also that coalition S finds the following strategy bundle $x''^S \in F^S(x)$:

$$x''^1(t) = x''^2(t) := \begin{cases} 1.5, & \text{if } t^2 = H^2, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that

$$E(x''^1 \mid H^1) = E(x''^1 \mid L^1) = 0.75 < 1 = E(x^1 \mid H^1) = E(x^1 \mid L^1),$$

so S cannot improve upon x using x''^S according to the traditional "blocking" criterion. However, when the true type profile is $\bar{t}^S = (H^1, H^2)$, player 2 wants to agree to the joint strategy x''^S. When this happens, player 1 infers that 2's true type is H^2, so 1 also wants to agree to x''^S. Thus, strategy x''^S serves as a "blocking" strategy against x^S. □

The heart of these examples lie in the fact that players are comparing two strategy bundles. In example 3.2.8, if the members of coalition S decide to form their coalition and adopt strategy bundle x''^S when the grand coalition has been deliberating on strategy bundle x, it is because they received the information that event $\{(H^1, H^2), (L^1, H^2)\}$ has realized, and both players are better off with x''^S than with x given this additional information. It is important to keep in mind that this kind of information revelation occurs within a "blocking" coalition. The scenario here (in which two strategy bundles x''^S and x are compared) does not address how the private information is revealed only through the original strategy bundle x of the grand coalition N. In particular, given a strong equilibrium strategy bundle (or a core strategy bundle) x^*, this kind of information revelation does not occur, since there are no "blocking" coalitions.

Notice that strategies x'^S and x''^S in examples 3.2.7 and 3.2.8 are not $\mathcal{T}^j$-measurable or Bayesian incentive-compatible. Ichiishi and Sertel (1998) noted that in the profit center game (example 2.2.3), information is not revealed through coalition formation: For any strategy $(x^S, y^S) \in \hat{F}^S$, $E(x^i \mid \mathcal{T}^i)$ is constant on T for every $i \in S$. Then a "blocking" strategy would have to make every $i \in S$ better off (in terms of the *interim* expected imputation) for all possible type profiles. Thus, the fact that division i joins a particular coalition does not reveal any information to the other divisions of the coalition.

3.2.3 By credible transmission of information during the contract negotiation

This approach, taken by Yazar (2001), endogenously determines a communication system as a part of coalition's strategy bundle during the *ex ante* period of strategy negotiation. Recall her formulation of a strategy in the Bayesian pure exchange economy (subsection 2.4.3) and her Bayesian incentive compatibility condition 2.4.7. Define for each coalition S the set of measurable, Bayesian incentive-compatible and attainable strategies:

$$\hat{F}^S := \left\{ \{z^j, \mathcal{C}^j\}_{j \in S} \;\middle|\; \begin{array}{l} \forall\, j \in S : z^j \text{ is } \vee_{i \in S}\, \mathcal{C}^i\text{-measurable, and } \mathcal{C}^j \subset \mathcal{T}^j, \\ \{z^j, \mathcal{C}^j\}_{j \in S} \text{ is Bayesian incentive-compatible,} \\ \forall\, t \in T : \sum_{j \in S} z^j(t) = \mathbf{0} \end{array} \right\}.$$

A strategy bundle $\{z^{*j}, \mathcal{C}^{*j}\}_{j \in N}$ of the grand coalition in Bayesian pure exchange economy $\mathcal{E}_{pe}$ is said to be in the *EC-core* (*endogenous communication plan core*), (i) if it is in $\hat{F}^N$, and (ii) if it is not true that there exist $S \in \mathcal{N}$ and $\{z^j, \mathcal{C}^j\}_{j \in S} \in \hat{F}^S$ such that $Eu^j(z^j + e^j) > Eu^j(z^{*j} + e^j)$ for every $j \in S$. The communication system $\{\mathcal{A}^{*j}\}_{j \in N}$, $\mathcal{A}^{*j} := \mathcal{T}^j \vee (\vee_{i \in N} \mathcal{C}^{*i})$, sustains as a result of credible talk at the contract negotiation.

Yazar's main result (theorem 3.2.10) follows immediately from her lemma on nested structures:

LEMMA 3.2.9 (Yazar, 2001) *For any coalition* $S \in \mathcal{N}$*, let* $\{\mathcal{C}^j\}_{j\in S}$ *and* $\{\mathcal{C}'^j\}_{j\in S}$ *be two communication plan bundles, and let* $\{z^j\}_{j\in S}$ *be a net trade bundle. If* $\mathcal{C}'^j \subset \mathcal{C}^j$ *for every* $j \in S$ *and if* $\{z^j, \mathcal{C}'^j\}_{j\in S} \in \hat{F}^S$*, then* $\{z^j, \mathcal{C}^j\}_{j\in S} \in \hat{F}^S$*.*

THEOREM 3.2.10 (Yazar, 2001) *Let* $\{\mathcal{C}^j\}_{j\in N}$ *and* $\{\mathcal{C}'^j\}_{j\in N}$ *be two communication plan bundles, and let* $\{z^j\}_{j\in N}$ *be a net trade bundle for the grand coalition. If* $\mathcal{C}'^j \subset \mathcal{C}^j$ *for every* $j \in N$ *and if* $\{z^j, \mathcal{C}'^j\}_{j\in N}$ *is in the EC-core, then* $\{z^j, \mathcal{C}^j\}_{j\in N}$ *is also in the EC-core.*

In particular, if the EC-core is nonempty at all, then there exists a strategy bundle in the EC-core which gives rise to the full communication system.

For the special case in which each utility function $u^j(\cdot, t)$ is affine linear on the consumption set $\mathbf{R}^l_+$, Yazar (2001) also established nonemptiness of the EC-core by direct application of Scarf's theorem for nonemptiness of the core (see Scarf (1973, theorem 8.3.6, p. 211)) to the non-side-payment game defined by

$$V(S) := \left\{ u \in \mathbf{R}^N \;\middle|\; \begin{array}{l} \exists\ \{z^j, \mathcal{C}^j\}_{j\in S} \in \hat{F}^S : \\ \forall\ t \in T : \sum_{j\in S} z^j(t) = \mathbf{0} \\ \forall\ j \in S : u_j \leq Eu^j(z^j + e^j) \end{array} \right\}.$$

For Vohra's Bayesian incentive compatibility (condition 2.4.5) applied to an arbitrary communication plan, the analogue of lemma 3.2.9 is trivially true. We may, therefore, assume *without loss of generality* that coalition S designs a strategy bundle $\{z^j, \mathcal{T}^j\}_{j\in S}$ with the full communication plan.

3.3 Efficiency

There are substantial amount of works on normative solution concepts for Bayesian cooperative games; for a survey, see, e.g., the introductory section of Rosenmüller (1992). In this subsection, we focus on a specific normative criterion, Pareto efficiency. Corresponding to the *interim* descriptive solution concepts of the coarse core (definition 2.5.1) and the fine core (definition 2.5.2), Wilson (1978) proposed two *interim* efficiency criteria (definitions 3.3.1 and 3.3.2 below) within the framework of Bayesian pure exchange economy (example 2.2.1) and examined these properties for several numerical examples. They can immediately extended to the Bayesian society.

DEFINITION 3.3.1 (Wilson, 1978) A commodity allocation plan x^* of Bayesian pure exchange economy $\mathcal{E}_{pe}$ is said to be *coarse efficient*, if
(i) $x^* \in F_T^N$; and
(ii) if it is not true that

$$\begin{aligned}&\exists\ E \in \wedge_{j\in N}\mathcal{T}^j : \exists\ x \in F_E^N :\\ &\qquad \forall\ j \in N : \forall\ t \in E : Eu^j(x^j \mid \mathcal{T}^j)(t) > Eu^j(x^{*j} \mid \mathcal{T}^j)(t).\end{aligned}$$

As the coalitional stability condition for the coarse core, the above efficiency condition (ii) is weak in the present extreme case of asymmetric information ($\mathcal{T}^i \wedge \mathcal{T}^j = \{\emptyset, T\}$ if $i \neq j$). Assuming $\#N \geq 2$, it is equivalent to:

$$\neg\ \exists\ x \in F_T^N : \forall\ j \in N : \forall\ t \in T : Eu^j(x^j \mid \mathcal{T}^j)(t) > Eu^j(x^{*j} \mid \mathcal{T}^j)(t);$$

it is weaker than *ex ante* efficiency.

DEFINITION 3.3.2 (Wilson, 1978) A commodity allocation plan x^* of Bayesian pure exchange economy $\mathcal{E}_{pe}$ is said to be *fine efficient*, if
(i) $x^* \in F_T^N$; and
(ii) if it is not true that

$$\begin{aligned}&\exists\ E \in \vee_{j\in N}\mathcal{T}^j : \exists\ x \in F_E^N :\\ &\quad \forall\ j \in N : \forall\ t \in E : \quad Eu^j(x^j \mid \vee_{i\in N}\mathcal{T}^i)(t) > Eu^j(x^{*j} \mid \vee_{i\in N}\mathcal{T}^i)(t).\end{aligned}$$

Notice that Wilson's fine efficiency corresponds to the situation in which the only available communication system is the full communication system. The above criterion (ii) in the present extreme case of asymmetric information amounts to the *ex post* efficiency criterion:

$$\neg\ \exists\ t \in T : \exists\ c \in F_{\{t\}}^N : \forall\ j \in N : u^j(c^j, t) > u^j(x^{*j}(t), t).$$

Hahn and Yannelis (1997) noted that many efficiency criteria can be defined (1) for different timings of evaluation (*ex ante*, *interim*, *ex post*) and (2) for different availabilities of information (coarse, private, fine). Among the various criteria they presented, the following is noteworthy:

DEFINITION 3.3.3 (Hahn and Yannelis, 1997) A commodity allocation plan x^* of Bayesian pure exchange economy $\mathcal{E}_{pe}$ is said to be *interim private efficient*, if
(i) $x^* \in F'^N$; and
(ii) if it is not true that

$$\exists\ t \in T : \exists\ x^N \in F'^N : \forall\ j \in N : Eu^j(x^j \mid t^j) > Eu^j(x^{*j} \mid t^j).$$

It is widely known that the Bayesian incentive compatibility is a major cause of *ex post* (Pareto-)inefficiency. Ichiishi and Sertel (1998) studied the *interim* Bayesian incentive-compatible core (definition 2.5.3) of the profit center game with incomplete information (example 2.2.3) and showed the following by an example: Recall the two-*interim*-period framework, the setup period and the manufacturing period. A full-information-revealing core plan reveals the complete information by the end of the setup period (that is, the manufacturing period becomes the *ex post* period). Players then realize the welfare loss created in the setup period. There are two sources of the loss, the information-revelation process and the Bayesian incentive compatibility. Ichiishi and Sertel (1998) introduced the new scenario that the divisions are free to make a re-contract after the setup period, provided that nobody will receive a lower utility level than was promised in the original contract. Such re-contract can remove the inefficiency caused by the Bayesian incentive compatibility. However, *ex post* inefficiency may still persist, because the divisions have made some of the choices at the time when each had no information about the others' types, that is, the loss caused by the information-revelation process is irreversible.

3.4 Comparisons of several core concepts

We first discuss comparison of the fine core and the *ex post* core. Einy, Moreno and Shotovitz (2000a) studied the Bayesian pure exchange economy with a nonatomic space of consumers, an incomplete-information version of Aumann's (1964) seminal model, and established that *a fine core allocation is an ex post core allocation*.

Actually, their comparison result is straightforward in our finite setup (example 2.2.1). We have already pointed out in subsection 2.5 that a fine core allocation x^* of the Bayesian pure exchange economy $\mathcal{E}_{pe}$ satisfies in particular,

$$\begin{array}{c}\neg\ \exists\ S \in \mathcal{N} : \exists\ t^S \in T^S : \exists\ x^S \in F^S_{\{t^S\}\times T^{N\setminus S}} : \\ \forall\ j \in S : \quad Eu^j(x^j \mid t^S) > Eu^j(x^{*j} \mid t^S).\end{array}$$

An *ex post core allocation* of economy $\mathcal{E}_{pe}$ is defined as a strategy bundle $x^* \in F^N_T$ such that

$$\neg\ \exists\ S \in \mathcal{N} : \exists\ t \in T : \exists\ c^S \in F^S_{\{t\}} : \forall\ j \in S : u^j(c^j, t) > u^j(x^{*j}(t), t).$$

The former coalitional stability condition (for the fine core) implies the latter coalitional stability condition (for the *ex post* core), if $\pi \gg \mathbf{0}$. Indeed, suppose the contrary,

$$\exists\ S \in \mathcal{N} : \exists\ t \in T : \exists\ c^S \in F^S_{\{t\}} : \forall\ j \in S : u^j(c^j, t) > u^j(x^{*j}(t), t).$$

Define $x^S \in F^S_{\{t^S\}\times T^{N\setminus S}}$ by

$$x^S(t^S, s^{N\setminus S}) := \begin{cases} c^S & \text{if} \quad s^{N\setminus S} = t^{N\setminus S} \\ x^{*S}(t^S, s^{N\setminus S}) & \text{if} \quad s^{N\setminus S} \neq t^{N\setminus S}. \end{cases}$$

Then,

$$\begin{aligned} Eu^j(x^j \mid t^S) &= \pi(t \mid t^S)u^j(c^j, t) \\ &\quad + \sum_{s^{N\setminus S}\neq t^{N\setminus S}} \pi(t^S, s^{N\setminus S} \mid t^S)u^j(x^{*S}(t^S, s^{N\setminus S}), (t^S, s^{N\setminus S})) \\ &> \pi(t \mid t^S)u^j(x^{*j}(t), t) \\ &\quad + \sum_{s^{N\setminus S}\neq t^{N\setminus S}} \pi(t^S, s^{N\setminus S} \mid t^S)u^j(x^{*S}(t^S, s^{N\setminus S}), (t^S, s^{N\setminus S})) \\ &= Eu^j(x^{*j} \mid t^S), \end{aligned}$$

violating the coalitional stability condition for the fine core. Thus, *every fine core allocation of the Bayesian pure exchange economy is an ex post core allocation.*

How this proposition is modified if we impose the basic requirements (measurability and Bayesian incentive compatibility) on the fine core is still an open question.

Krasa and Shafer (2001) formulated incomplete information differently from the present type-profile framework (subsection 2.1), and defined the measurability requirement (c.f. subsection 2.3) and the Bayesian incentive compatibility requirement (c.f. subsection 2.4) within their own framework. They applied these requirements to the pure exchange economy in order to define the private core and the Bayesian incentive-compatible core, respectively. The incentive-compatible core may not satisfy the measurability condition here. According to their formulation, the concept of convergence of incomplete information to the complete information is well-defined. Roughly stated, their results are that *in general the private core allocations does not converge to any complete information core allocation as the incomplete information converges to the complete information*, but that *almost all complete information core allocations can be the limit of incentive-compatible core allocations as the incomplete information converges to the complete information.* While their formulation cannot encompass the one-shot model of the present paper (subsection 2.1), their results nevertheless provide insights into the nature of the two requirements.

We present Krasa and Shafer's model now. Let N be a finite set of consumers, and let Ω be a *state space*. In the case of the Bayesian pure exchange economy (example 2.2.1), the state space is given as $\Omega = \prod_{j \in N} T^j$, but in the Krasa-Shafer formulation, the space Ω is arbitrarily given. Incomplete information is defined as a noisy *signal*: when state $\omega \in \Omega$ is realized, player j receives the wrong information with a positive probability that state $\omega' \in \Omega \setminus \{\omega\}$ has occurred. Let $\Phi^j := \Omega$, player j's signal space. Let $\Phi := \prod_{j \in N} \Phi^j$, the signal bundle space. *Incomplete information* is defined as a probability π on $\Omega \times \Phi$. In the private information case, consumer j observes signal $\omega' \in \Phi^j$ and infers that event $E \subset \Omega$ has occurred with probability

$$\frac{\pi(E \times \Omega \times \cdots \times \{\omega'\} \times \cdots \times \Omega)}{\pi(\Omega \times \Omega \times \cdots \times \{\omega'\} \times \cdots \times \Omega)}.$$

Complete information is defined as a probability $\hat{\pi}$ on $\Omega \times \Phi$ such that $\hat{\pi}(\{(\omega, \omega, \cdots, \omega) \in \Omega \times \Phi \mid \omega \in \Omega\}) = 1$. A sequence of incomplete information is said to converge to a complete information, if $\pi \to \hat{\pi}$. The complete information $\hat{\pi}$ defined in the Krasa-Shafer framework is different from the complete information defined in the type-profile framework of subsection 2.1; see remark 3.4.1 below.

The consumption set of each consumer is the nonnegative orthant $\mathbf{R}^l_+$ of the commodity space $\mathbf{R}^l$. Player j's preference relation is represented by a state-dependent von Neumann-Morgenstern utility function, $u^j : \mathbf{R}^l_+ \times \Omega \to \mathbf{R}$. His initial endowment is a state-dependent commodity bundle, $e^j : \Omega \to \mathbf{R}^l_+$. The *K-S pure exchange economy* is thus given as a list of specified data, $\mathcal{E}_{ks} := \Big(\Omega, \pi, \{\mathbf{R}^l_+, u^j, e^j\}_{j \in N}\Big)$.

Consumer j's strategy is a (state, signal bundle)-contingent commodity bundle, $x^j : \Omega \times \Phi \to \mathbf{R}^l_+$. Its *ex ante* expected utility is

$$Eu^j(x^j) := \int_{\Omega \times \Phi} u^j(x^j(\omega, \phi), \omega) d\pi(\omega, \phi).$$

Let $\hat{\pi}$ be a complete information. A *complete information core allocation* of a K-S pure exchange economy (with $\hat{\pi}$) is a strategy bundle $\{x^{*j}\}_{j \in N}$ such that it is feasible in the grand coalition:

$$\sum_{j \in N} x^{*j}(\omega, \omega, \cdots, \omega) = \sum_{j \in N} e^j(\omega), \quad \hat{\pi}\text{-a.e.},$$

and such that it is not improved upon by any coalition:

$$\begin{aligned} &\neg\, \exists\, S \in \mathcal{N} : \exists\, x^S : \Omega \times \Phi \to \mathbf{R}^{l \cdot \# S}_+ \\ &\qquad \sum_{j \in S} x^j(\omega, \omega, \cdots, \omega) = \sum_{j \in S} e^j(\omega), \quad \hat{\pi}\text{-a.e.} \\ &\qquad \forall\, j \in S : Eu^j(x^j) > Eu^j(x^{*j}), \end{aligned}$$

where the expectation is taken with respect to the complete information $\hat{\pi}$.

The private information structure $\mathcal{T}^j$ of player j is the algebra on $\Omega \times \Phi$, generated by the sets,

$$\begin{aligned} &\{\omega \in \Omega \mid e^j(\omega) = e\} \times \Phi, \quad e \in \mathbf{R}^l_+, \quad \text{and} \\ &\Omega \times \Omega \times \cdots \times \{\omega\} \times \cdots \times \Omega, \quad \omega \in \Phi^j. \end{aligned}$$

The private information structure $\mathcal{T}^j$ defined in the Krasa-Shafer framework is different from the private information structure defined in the type-profile framework of subsection 2.1; see remark 3.4.1 below.

Given a (general) incomplete information π, a *private core allocation* of a K-S pure exchange economy (with π) is defined exactly as the above complete information core allocation, except that all strategies, x^{*j} in the grand coalition and x^j in the blocking coalition, are $\mathcal{T}^j$-measurable.

Consumer j's strategy $x^j : \Omega \times \Phi \to \mathbf{R}^l_+$ is called *Bayesian incentive-compatible*, if misrepresentation of his signal does not increase his *ex ante* expected utility:,

$$\begin{aligned}
&\forall\ \phi = \{\phi^i\}_{i \in N} \in \Phi : \forall\ \phi'^j \in \Phi^j : \\
&\int_{\Omega \times \Phi^{N \setminus \{j\}}} u^j(x^j(\omega, \phi), \omega) d\pi(\omega, \phi^{N \setminus \{j\}} \mid \phi^j, e^j) \\
&\quad \geq \int_{\Omega \times \Phi^{N \setminus \{j\}}} u^j(x^j(\omega, \phi^{N \setminus \{j\}}, \phi'^j), \omega) d\pi(\omega, \phi^{N \setminus \{j\}} \mid \phi^j, e^j).
\end{aligned}$$

Given a (general) incomplete information π, a *Bayesian incentive-compatible core allocation* of a K-S pure exchange economy (with π) is defined exactly as the above complete information core allocation, except that all strategies, x^{*j} in the grand coalition and x^j in the blocking coalition, are Bayesian incentive-compatible. It does not have to be $\mathcal{T}^j$-measurable.

REMARK 3.4.1 The complete information $\hat{\pi}$ and the private information structure $\mathcal{T}^j$ defined in the Krasa-Shafer framework are different from those defined in the type-profile framework of subsection 2.1. According to the latter definition of the complete information, the *ex ante* stage and the *ex post* stage are identical, that is, $\#\Omega = 1$, or almost equivalently (in the case $\#\Omega$ is arbitrary), there exists $\omega \in \Omega$ which every player knows will occur surely ($\hat{\pi}(\omega, \omega, \cdots, \omega) = 1$).

In the Krasa-Shafer framework, let $\pi(\cdot \mid \phi^j)$ be the conditional probability of π given $\phi^j \in \Phi^j$. Corresponding to the assumption of subsection 2.1 that the *ex ante* probability on the type-profile space is strictly positive, assume here that the support of $\pi(\cdot \mid \phi^j)$ is of the form,

$$\text{supp } \pi(\cdot \mid \phi^j) = E \times \Omega \times \cdots \times \Omega.$$

Then the algebra on $\Omega \times \Phi$ generated by the sets,

$$\{(\omega, \phi) \in \Omega \times \Phi \mid (\omega, \phi^{N \setminus \{j\}}) \in \text{supp } \pi(\cdot \mid \bar{\phi}^j), \phi^j = \bar{\phi}^j\},\ \bar{\phi}^j \in \Phi^j,$$

is the private information structure defined in subsection 2.1.

We demonstrate by an example how Krasa and Shafer's framework cannot encompass our model of Bayesian pure exchange economy (example 2.2.1). Consider the Bayesian pure exchange economy $\mathcal{E}_{pe}$ with two consumers ($N =$

$\{1,2\}$), such that consumer 1 has one type and consumer 2 has two types ($T^1 = \{t^1\}$, $T^2 = \{t^2_a, t^2_b\}$). We write for simplicity,

$$T = \{(t^1, t^2_a), (t^1, t^2_b)\} =: \{a, b\}.$$

The objective *ex ante* probability on T is given by the probability which assigns density p on $\{a\}$, and $(1-p)$ on $\{b\}$.

Krasa and Shafer's state space Ω is the type-profile space T in the present model $\mathcal{E}_{pe}$, so the complete information π (*ex ante* probability on $\Omega \times \Phi := T \times T \times T$) is given by the condition,

$$\pi(\{(a,a,a),(b,b,b)\}) = 1.$$

On the other hand, the incomplete information π_p on $\Omega \times \Phi$ derived from the probability $(p, 1-p)$ on T is

$$\pi_p(\omega, \phi^1, \phi^2) = \begin{cases} p^2 & \text{if} \quad (\omega, \phi^1, \phi^2) = (a,a,a), \\ 0 & \text{if} \quad (\omega, \phi^1, \phi^2) = (a,a,b), \\ p(1-p) & \text{if} \quad (\omega, \phi^1, \phi^2) = (a,b,a), \\ 0 & \text{if} \quad (\omega, \phi^1, \phi^2) = (a,b,b), \\ 0 & \text{if} \quad (\omega, \phi^1, \phi^2) = (b,a,a), \\ (1-p)p & \text{if} \quad (\omega, \phi^1, \phi^2) = (b,a,b), \\ 0 & \text{if} \quad (\omega, \phi^1, \phi^2) = (b,b,a), \\ (1-p)^2 & \text{if} \quad (\omega, \phi^1, \phi^2) = (b,b,b), \end{cases}$$

because player 1 has probability $(p, 1-p)$ on T at the *interim* stage, which is uncorrelated with the *ex ante* probability $(p, 1-p)$ on the state space T, and player 2 has the complete information at the *interim* stage. Thus, as long as the complete information π assigns a strictly positive probability on every point (that is, $\pi(\omega, \omega, \omega) > 0$ for each $\omega \in \Omega$), π_p cannot converge to π no matter how p behaves.

When $p \to 1$, the incomplete information π_p does converge to the specific complete information $\hat{\pi}$, given by $\hat{\pi}(a,a,a) = 1$. The limit probability $\hat{\pi}$ says that the *ex ante* state is identical to the *ex post* state, i.e., the sure occurrence of $a \in \Omega$; this situation is much stronger than the Krasa-Shafer definition of complete information. $\square$

3.5 Pure exchange economy

To date many works have been done on the Bayesian pure exchange economy (example 2.2.1),

$$\mathcal{E}_{pe} := \left(\{C^j, T^j, u^j, e^j\}_{j\in N},\ \pi\right),$$

where set C^j is consumer j's consumption set. These works are mostly on the existence of a core allocation, and on the core convergence theorem. We assume that either each consumption set is the nonnegative orthant $\mathbf{R}^l_+$ of the commodity space, or it is a nonempty and compact subset of $\mathbf{R}^l$ (the latter assumption can be made without loss of generality).

3.5.1 Existence

We have already reviewed some existence results for the Bayesian pure exchange economy in subsection 3.1, since they can straightforwardly be extended to the general model of Bayesian society as strong equilibrium existence theorems. We present here further works on the existence of a core allocation: Vohra's (1999) Bayesian incentive-compatible coarse core allocation existence theorem for the mediator-base approach, two immediate consequences of Ichiishi and Idzik's (1996) general existence result on an *ex ante* solution (theorem 3.1.2), and an *ex ante* Bayesian incentive-compatible core allocation existence theorem for the private information case. It is unlikely that Vohra's (1999) result and the *ex ante* existence theorem for the private information case can be extended beyond the Bayesian pure exchange economy.

Vohra (1999, proposition 3.1) provided a sufficient condition for the existence of a Bayesian incentive-compatible coarse core allocation of the Bayesian pure exchange economy $\mathcal{E}_{pe}$ for the mediator-based approach. In his framework, use of a net trade plan $z^j : T \to \mathbf{R}^l$ as player j's strategy (instead of a demand plan $t \mapsto z^j(t) + e^j(t^j)$) is crucial. Also crucial is his assumption that j's *ex ante* probability π^j takes value 0 on some type profiles, and the support of π^i and the support of π^j are the same for all $i, j \in N$. Denote by T^* the support of π^j; $T^* := \{t \in T \mid \pi^j(t) > 0\}$. Information is called *non-exclusive*, if unilateral deception can be detected, that is, if

$$\forall\ t \in T^* : \forall\ j \in N : \forall\ s^j \in T^j \setminus \{t^j\} : \pi^j(s^j, t^{N\setminus\{j\}}) = 0.$$

LEMMA 3.5.1 (Vohra 1999) *Let $\mathcal{E}_{pe}$ be the Bayesian pure exchange economy, in which information is non-exclusive and each consumer j's strategy is his net trade plan. Let $z : T \to \mathbf{R}^{l \cdot \# N}$ be attainable net trade plans ($\{z^j + e^j\}_{j \in N} \in F_T^N$) such that $Eu^j(z^j + e^j \mid t^j) \geq Eu^j(e^j \mid t^j)$ for every $j \in N$ and $t^j \in T^j$. Then there exists an attainable and Bayesian incentive-compatible net trade plans $\hat{z} : T \to \mathbf{R}^{l \cdot \# N}$ such that z and $\hat{z}$ give rise to the same interim expected utility allocation,*

$$\forall\ j \in N : \forall\ t^j \in T^j : Eu^j(z^j + e^j \mid t^j) = Eu^j(\hat{z}^j + e^j \mid t^j).$$

Proof Let z be the net trade plans given in the lemma. Define $\hat{z} : T \to \mathbf{R}^{l \cdot \# N}$ by

$$\hat{z}^j(t) := \begin{cases} z^j(t) & \text{if } t \in T^*, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\hat{z}$ is attainable, and gives rise to the same interim expected utility allocation as z. We only need to check the Bayesian incentive compatibility. Choose any $t^j \in T^j$ and any $s^j \in T^j \setminus \{t^j\}$.

$$\begin{aligned}
& Eu^j(\hat{z}^j(s^j, \cdot) + e^j(t^j) \mid t^j) \\
= \;& \sum_{\tau^{N \setminus \{j\}} \in T^{N \setminus \{j\}}} \pi^j(\tau^{N \setminus \{j\}} \mid t^j) u^j(\hat{z}^j(s^j, \tau^{N \setminus \{j\}}) + e^j(t^j)) \\
= \;& \sum_{\tau^{N \setminus \{j\}} \in T^{N \setminus \{j\}} : (t^j, \tau^{N \setminus \{j\}}) \in T^*} \pi^j(\tau^{N \setminus \{j\}} \mid t^j) u^j(\hat{z}^j(s^j, \tau^{N \setminus \{j\}}) + e^j(t^j)) \\
= \;& \sum_{\tau^{N \setminus \{j\}} \in T^{N \setminus \{j\}} : (t^j, \tau^{N \setminus \{j\}}) \in T^*} \pi^j(\tau^{N \setminus \{j\}} \mid t^j) u^j(e^j(t^j)) \\
\leq \;& Eu^j(z^j + e^j \mid t^j) \\
= \;& Eu^j(\hat{z}^j + e^j \mid t^j).
\end{aligned}$$

□

Proof of Vohra's existence theorem is completed by combining Wilson's existence theorem and lemma 3.5.1 as follows: Let z be a coarse core allocation, whose existence is asserted by Wilson. In the light of lemma 3.5.1, there exists an attainable, Bayesian incentive-compatible allocation $\hat{z}$ which gives rise to the same *interim* expected utility allocation as z. No coalition can improve upon z using its attainable allocation, Bayesian incentive-compatible or not. So no coalition can improve upon $\hat{z}$ using its Bayesian incentive-compatible, attainable allocation.

Non-exclusive information is illustrated in figure 1: the Vohra box diagram. Here, $N = \{1, 2, 3\}$, and each type space has two elements, $T^j = \{t_1^j, t_2^j\}$, $j \in N$. The support of each *ex ante* probability is given as

$$T^* = \left\{ (t_1^1, t_1^2, t_2^3),\ (t_2^1, t_1^2, t_1^3),\ (t_2^1, t_2^2, t_2^3) \right\},$$

indicated by the shaded area in figure 1. Notice that for each type profile $t \in T^*$, there exists one player who knows the precise realization of t. Information is obtained not only through the private information structure but also through the marginal probability.

Vohra's (1999) main result is an example of a Bayesian pure exchange economy for the mediator-based approach whose Bayesian incentive-compatible coarse core is empty (example 3.2, pp. 136-137). Needless to say, he constructs nonlinear utility functions.

The first consequence of Ichiishi and Idzik (1996) is an obvious special case: If each utility function $u^j(\cdot, t)$ of economy $\mathcal{E}_{pe}$ is affine linear on C^j, then there exists a Bayesian incentive-compatible core allocation.

The second consequence, which is also straightforward, is on economy $\mathcal{E}_{pe}$ in which everybody takes a mixed strategy (or rather, type-profile dependent mixed choices) and the coalitional attainability is defined as the expected feasiblity. Consumer j's choice set is now the set $\mathcal{M}(C^j)$ of all probabilities on the compact consumption set C^j. His strategy is a function $\mu^j : T \to \mathcal{M}(C^j)$, $t \mapsto \mu^j[t](\cdot)$; denote by X^j the set of all such functions. His preference relation is represented by the expected utility function $U^j : \mathcal{M}(C^j) \times T \to \mathbf{R}$ defined by

$$U^j(p^j, t) := \int_{C^j} u(c^j, t) p^j(dc^j).$$

Coalition S's feasible strategy bundle is a $\mathcal{T}^S$-measurable member of X^S such that for each t the expected total demand is less than or equal to the total supply. Coalition S's feasible strategy correspondence is the constant correspondence on X that takes the value,

$$F^S := \left\{ \mu^S : T \to \prod_{j \in S} \mathcal{M}(C^j) \;\middle|\; \begin{array}{l} \mu^S \text{ is } \mathcal{T}^S\text{-measurable,} \\ \forall\ t : \sum_{j \in S} \int_{C^j} c^j \mu^j[t](dc^j) \leq \sum_{j \in S} e^j(t) \end{array} \right\}.$$

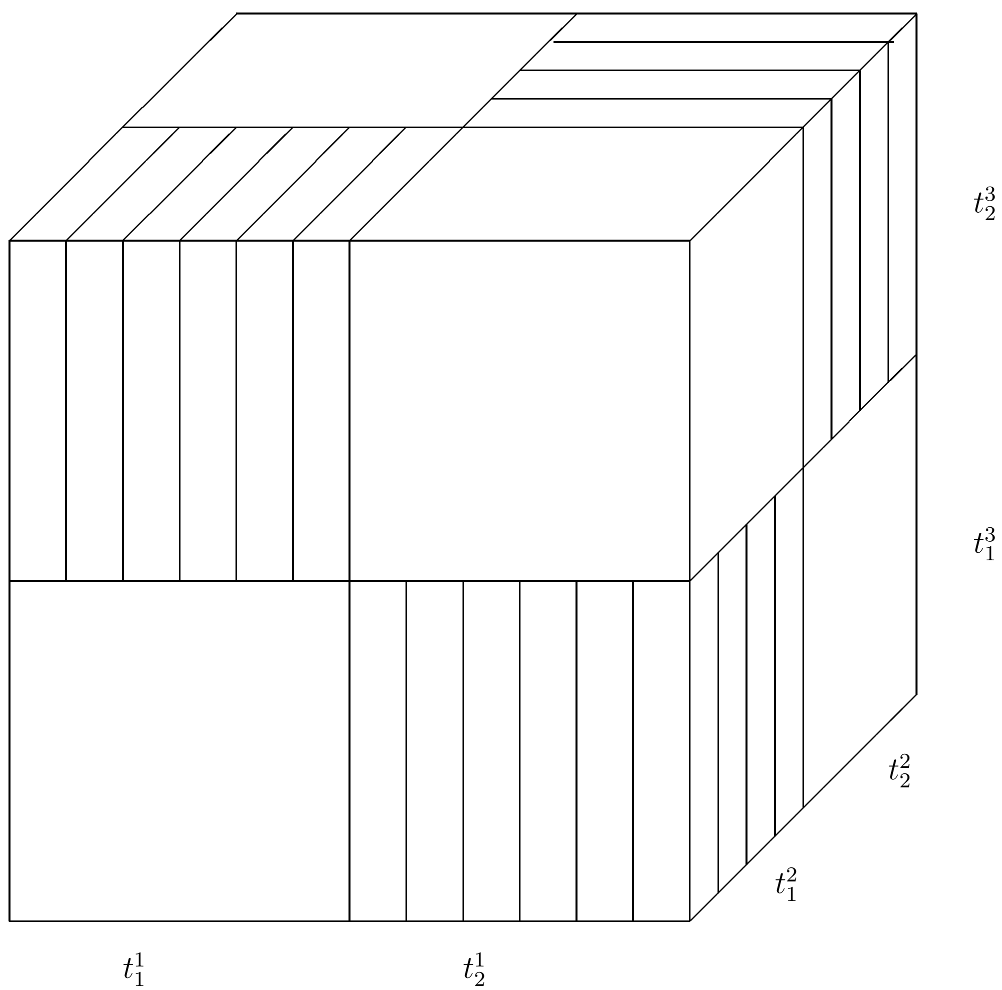

Figure 1: The Vohra box diagram

We thus have the associated Bayesian society,

$$\mathcal{S} := \Big(\{\mathcal{M}(C^j), T^j, U^j\}_{j\in N},\ \{F^S\}_{S\in\mathcal{N}},\ \pi\Big).$$

COROLLARY 3.5.2 *Let $\mathcal{E}_{pe}$ be a Bayesian pure exchange economy, in which each consumption set C^j is a nonempty and compact subset of the commodity space $\mathbf{R}^l$ and each von Neumann-Morgenstern utility function $u^j(\cdot,t)$ is continuous in C^j. Allow each consumer to take a mixed choice, and define the coalitional attainability as the expected feasibility at each type profile. Then there exists a Bayesian incentive-compatible core allocation.*

Proof We only have to verify that conditions (i)-(vi) of theorem 3.1.2 are all satisfied in the associate Bayesian society.

(i) The choice set $\mathcal{M}(C^j)$ endowed with the weak* topology is compact, convex, and metrizable.

(ii) Clearly, for any j and any t, $U^j(\cdot,t)$ is linear affine and continuous in $(\mathcal{M}(C^j)$, weak* topology).

(iv) and (vi) Correspondence $t \mapsto F^S$, being a constant correspondence, is both upper semicontinuous and lower semicontinuous. Set F^S is clearly nonempty, closed and convex.

(v) Let $\mathcal{B}$ be a balanced family of subsets of N with associated balancing coefficients $\{\lambda_S\}_{S\in\mathcal{B}}$. Choose any $\{\mu^{S,j}\}_{j\in S} \in F^S$ and define $\nu^j := \sum_{S\in\mathcal{B}:S\ni j}\lambda_S\mu^{S,j}$ for every $j\in N$. Then,

$$\begin{aligned}\sum_{j\in N}\int_{C^j} c^j\nu^j[t](dc^j) &= \sum_{j\in N}\sum_{S\in\mathcal{B}:S\ni j}\lambda_S\int_{C^j} c^j\mu^{S,j}[t](dc^j)\\ &= \sum_{S\in\mathcal{B}}\lambda_S\sum_{j\in S}\int_{C^j} c^j\mu^{S,j}[t](dc^j)\\ &\le \sum_{S\in\mathcal{B}}\lambda_S\sum_{j\in S} e^j(t)\\ &= \sum_{j\in N} e^j(t).\end{aligned}$$

So $\nu \in F^N$. □

We have reviewed that affine linearity of j's utility function on C^j plays an essential role in establishing the existence of a Bayesian incentive compatible core allocation in the private information case (see theorem 3.1.2 and the sketch of its proof). This point is valid specifically in the Bayesian pure exchange economy when each consumer chooses a demand plan as his strategy. However, the story is different in the situation in which each consumer chooses a net trade plan as his strategy; see Forges, Minelli and Vohra (2000, proposition 1) for the following theorem.

THEOREM 3.5.3 *Let $\mathcal{E}_{pe}$ be a Bayesian pure exchange economy, in which each consumption set C^j is the nonnegative orthant $\mathbf{R}^l_+$ of the commodity space, and each von Neumann-Morgenstern utility function $u^j(\cdot, t)$ is concave, weakly monotone and continuous in C^j. Suppose that each consumer chooses a net trade plan as a strategy. Then there exists a Bayesian incentive-compatible core allocation.*

Proof Define a non-side-payment game $V : \mathcal{N} \to \mathbf{R}^N$ by

$$V(S) := \left\{ u \in \mathbf{R}^N \;\middle|\; \begin{array}{l} \exists\ z^S : T \to \mathbf{R}^{l \cdot \# S} : \\ \forall\ j \in S : z^j \text{ is } \mathcal{T}^j\text{-measurable}, \\ \forall\ t \in T : z^j(t^j) + e^j(t^j) \in \mathbf{R}^l_+,\ \sum_{j \in S} z^j(t^j) \leq \mathbf{0}, \\ \forall\ j \in S : u_j \leq E u^j(z^j + e^j) \end{array} \right\}.$$

The sets $V(S)$ are nonempty; indeed, the no-trade strategy bundle, $z^S(t) \equiv \mathbf{0}$, is always feasible, so gives rise to a member of $V(S)$. By direct application of Scarf's theorem for nonemptiness of the core (see, e.g., Scarf (1973, theorem 8.3.6, p. 211)), the core of this game V is nonempty. Let $z^\dagger$ be a net trade plan bundle which gives rise to a member of the core. By lemma 6.3 of Ichiishi and Radner (1999), re-produced in this paper as lemma 3.2.6, there exists a net trade plan bundle z^* such that each z^{*j} is $\mathcal{T}^j$-measurable, $z^{\dagger j} \leq z^{*j}$, and $\sum_{j \in N} z^{*j}(t^j) = \mathbf{0}$, for all $t \in T$. By weak monotonicity of $u^j(\cdot, t)$, the bundle z^* gives rise to a member of the core of V. By proposition 6.10 of Hahn and Yannelis (1997), re-produced in this paper as proposition 2.4.3, each strategy z^{*j} is Bayesian incentive-compatible. □

REMARK 3.5.4 Forges and Minelli (2001) took the mediator-based approach, and constructed another kind of Bayesian pure exchange economy with probabilistic choices. Let $\mathcal{E}_{pe}$ be the deterministic Bayesian pure exchange economy (example 2.2.1). For each type profile t, let $\mathbf{C}^S_0(t)$ be the compact set of commodity bundles attainable in coalition S,

$$\mathbf{C}^S_0(t) := \left\{ c^S \in C^S \;\middle|\; \sum_{j \in S} c^j \leq \sum_{j \in S} e^j(t) \right\},$$

and denote by $\mathcal{M}(\mathbf{C}_0^S(t))$ the set of all probabilities on $\mathbf{C}_0^S(t)$. An element of $\mathcal{M}(\mathbf{C}_0^S(t))$ is not *mixed* choices, but a *correlated* choice. The product probability of mixed strategies is a correlated strategy, but in general a correlated strategy cannot be expressed as the product probability of mixed strategies. A coalitional feasible strategy is defined as a selection $\mu^S : t \mapsto \mu^S[t](\cdot)$ of the correspondence, $t \mapsto \mathcal{M}(\mathbf{C}_0^S(t))$. Given coalitional strategy μ^S, consumer j's utility at the true type profile $\bar{t}$ is

$$U^j(\mu^S[\bar{t}], \bar{t}) := \int_{C^j} u^j(c^j, \bar{t})\mu^S[\bar{t}](dc).$$

In this correlated choice framework, individuals cannot singly choose *his* probability on commodity bundles. His "action" is defined as a report of his type. If consumer j misrepresent his type as $\tilde{t}^j$, assuming that everybody else supplies honest reports, j's utility becomes

$$U^j(\mu^S[\tilde{t}^j, \bar{t}^{N\setminus\{j\}}], \bar{t}) := \int_{C^j} u^j(c^j, \bar{t})\mu^S[\tilde{t}^j, \bar{t}^{N\setminus\{j\}}](dc).$$

A coalitional feasible strategy μ^S is called Bayesian incentive-compatible, if nobody benefits from misrepresenting his own type, that is,

$$\forall\ \bar{t} \in T : \forall\ j \in S : \forall\ \tilde{t}^j \in T^j : U^j(\mu^S[\bar{t}], \bar{t}) \geq U^j(\mu^S[\tilde{t}^j, \bar{t}^{N\setminus\{j\}}], \bar{t}).$$

Consumer j's *ex ante* expected utility of coalitional feasible strategy μ^S is given as $EU^j(\mu^S) := \sum_t U^j(\mu^S[t], t)\pi(t)$, and from this we can define the *ex ante* Bayesian incentive-compatible core.

Forges and Minelli (2001) established in this framework that *if $l = \#N$, each utility function $u^j(\cdot, t)$ is additively separable in C^j, e^j is a constant function, and $e^j_h = 0$ for all $h \neq j$, then the ex ante Bayesian incentive-compatible core is nonempty.*

In the absence of Forges and Minelli's restrictive assumptions, we expect that the core may be empty. While their model does not naturally fit in our model of Bayesian society (definition 2.1.3) due to the use of correlated choices, we can nevertheless imbed it in a particular Bayesian society: Redefine coalition S's feasible strategy set as

$$F^S := \left\{ \{\mu^j\}_{j\in S} \text{ defined on } T \;\middle|\; \begin{array}{l} \forall\ j \in S : \mu^j[t](\cdot) \in \mathcal{M}(\mathbf{C}_0^S(t)) \\ \forall\ i, j \in S : \mu^i[t](\cdot) = \mu^j[t](\cdot) \end{array} \right\}.$$

When the members take a strategy bundle $\{\mu^j\}_{j\in S} \in F^S$, everybody is choosing the same strategy, and this common strategy has been called a coalitional strategy, denoted by μ^S. We view that C^S is a subset of C^N (by setting the coordinates corresponding to $N\setminus S$ to be equal to $\mathbf{0}$). In the Bayesian society

$$\Big(\{\mathcal{M}(C^N), T^j, U^j\}_{j\in N},\ \{F^S\}_{S\in\mathcal{N}},\ \pi\Big)$$

constructed this way, assumptions (i)-(iv) and (vi) are satisfied (apart from the $\mathcal{T}^j$-measurability requirement, whose omission does not change the theorem as pointed out in remark 3.1.3). But assumption (v) is violated, hence the possibility of empty Bayesian incentive-compatible core. To see how (v) is violated, let $\mathcal{B}$ be a balanced family of subsets of N, and let $\{\lambda_S\}_{S\in\mathcal{B}}$ be the associated balancing coefficients. For each $S \in \mathcal{B}$, choose $\{\mu^S\}_{j\in S} \in F^S$ and define $\{\nu^j\}_{j\in N}$ by $\nu^j := \sum_{S\in\mathcal{B}:S\ni j} \lambda_S \mu^S$. Then $\nu^i \neq \nu^j$ for some $i \neq j$, so $\{\nu^j\}_{j\in N} \notin F^N$. Indeed, Forges, Mertens and Vohra (2000) further specified the model by introducing "money" and by assuming that each utility function is linear in money, and provided an example which has no Bayesian incentive-compatible core.

We point out two problems concerning the correlated-choice approach: First, like the mixed-choice approach or more generally like any probabilistic approach, it avoids the question of explaining deterministic choice. Even when a player has decided on a probabilistic choice, a time will come when he has to take a definite action. Usually in real life, he acts upon his own will (which theory needs to explain), and does not leave his action up to the outcome of throwing dice.

The second problem concerns specifically the correlated-choice approach. The approach is applicable in general only to situations in which outsiders to a coalition have no influence on the insiders, like the pure exchange economy (example 2.2.1) and the coalition production economy (example 2.2.2). It is not applicable to situations in which a coalition's feasible-strategy set depends on an outsiders' strategy bundle, or a player's utility depends on a choice bundle. To see this point, consider the simple no-externality case addressed by theorem 3.1.2, re-formulated by introducing type-profile dependent correlated choices as strategies. Suppose the grand coalition is choosing a strategy $t \mapsto \mu^N[t](\cdot)$, and coalition S is going to defect. In accordance with the spirit of the strong equilibrium, the members of S passively take the outsiders' strategies as given. However, there is no way to identify the part of the strategy μ^N that the outsiders are responsible for, so the members of S do not know which strategy of $N \setminus S$ they can passively take. One might argue that S takes the marginal probability $\mathrm{proj}_{N\setminus S}\ \mu^N[t] \in \mathcal{M}(C^{N\setminus S})$ as given. Yet there is no guarantee that the product of the two marginal probabilities, $\mathrm{proj}_{N\setminus S}\ \mu^N[t]$ and $\mathrm{proj}_S\ \mu^N[t]$, can recover the original probability, $\mu^N[t]$. The same problem occurs when a player's utility depends fully on a choice bundle. Thus, the approach fails to address intercoalitional problems in which several coalitions influence each other: the situations commonly observed in the present-day economy with organizations.

Instead of Forges and Minelli's (2001) correlated choices, we can use randomized choices defined as functions $f^j : P \to C^j$ for some probability space $(P, \mathcal{P}, p)$. From a randomized choice bundle $f^S := \{f^j\}_{j\in S}$, we derive a correlated choice as its distribution $p\circ(f^S)^{-1}$. An individual choice is re-defined as this randomized choice. This model can address the general case with externalities (although the first problem about the very moment of definite action still remains). We expect that an *ex ante* Bayesian incentive-compatible core allocation existence theorem can be established for a Bayesian pure exchange economy with randomized choices in the same way as theorem 3.1.2. $\square$

3.5.2 Core convergence/equivalence theorems

We turn to a bulk of works on the Edgeworth conjecture (the core convergence theorem or the core equivalence theorem) for the Bayesian pure exchange economy. In the course of studying this issue, various competitive equilibrium concepts have been invoked or newly proposed; some suffer from conceptual difficulties. We believe, however, that a core convergence/equivalence result is meaningful only if it approximates/characterizes a competitive equilibrium which is defined sensibly enough so that one expects to realize in the competitive market.

We have pointed out that Einy, Moreno and Shotovitz (2000a) established within the framework of a nonatomic space of consumers that a fine core allocation is an *ex post* core allocation. Notice that the *ex post* stage is essentially the complete information stage. They invoked Aumann's (1964) equivalence theorem for the pure exchange economy with complete information, and asserted as a corollary that *a fine core allocation of the Bayesian pure exchange economy with a nonatomic measure space of consumers is an ex post competitive allocation.*

Forges, Heifetz and Minelli (2001) studied the Bayesian pure exchange economy $\mathcal{E}_{pe}$, in which each initial endowment is a constant function, $T \to \mathbf{R}^l$, $t \mapsto e^j$, and in which everybody j in coalition S chooses a mixed strategy (or rather, type-profile dependent mixed choices) $\mu^j : T^S \to \mathcal{M}(C^j)$, and the coalitional attainability is defined as the *ex ante* expected feasibility,

$$\sum_{j\in S}\sum_{t^S\in T^S} \pi(\{t^S\}\times T^{N\setminus S}) \int_{C^j} c^j \mu^j[t^S](dc^j) \leq \sum_{j\in S} e^j,$$

that is, the market clearance on the average across type profiles as well as across pure choices (compare with the attainability definition in corollary 3.5.2 as the expected feasibility at each type profile). Notice that the *ex ante* expected feasibility does not guarantee the expected feasibility at the *interim* time of strategy execution. They took the mediator-based approach, and considered the Bayesian incentive compatibility (see condition 2.4.5),

$$\begin{aligned}&\forall\ \bar{t}^j,\ \tilde{t}^j \in T^j :\\ &\qquad \sum_{t\in T} \pi(t \mid \bar{t}^j) U^j(\mu^j[t^S], t) \geq \sum_{t\in T} \pi(t \mid \bar{t}^j) U^j(\mu^j[\tilde{t}^j, t^{S\setminus\{j\}}], t),\end{aligned}$$

where

$$U^j(\mu^j[\tilde{t}^j, t^{S\setminus\{j\}}], t) := \int_{C^j} u^j(c^j, t)\mu^j[\tilde{t}^j, t^{S\setminus\{j\}}](dc^j).$$

Denote by $F^{ic,S}$ the set of all strategy bundles of coalition S that satisfy the *ex ante* expected feasibility and the Bayesian incentive compatibility. Set for simplicity, $F^{ic} := F^{ic,N}$. The *ex ante* expected utility of strategy μ^j is given as $EU^j(\mu^j) := \sum_{t\in T} \pi(t) U^j(\mu^j[t^S], t)$.

Forges, Heifetz and Minelli defined an *ex ante core allocation* as a strategy bundle $\{\mu^{*j}\}_{j\in N}$, such that (i) it is attainable in the grand coalition,

$$\{\mu^{*j}\}_{j\in N} \in F^{ic},$$

and (ii) it is not weakly improved upon by any coalition,

$$\begin{aligned}\neg\ \exists\ S \in \mathcal{N} : \exists\ \{\mu^j\}_{j\in S} \in F^{ic,S} :\ \ &\forall\ j \in S :\ \ EU^j(\mu^j) \geq EU^j(\mu^{*j}),\\ &\exists\ j \in S :\ \ EU^j(\mu^j) > EU^j(\mu^{*j}).\end{aligned}$$

They defined an *ex ante competitive equilibrium* as a pair of a price vector $p^* \in \mathbf{R}^l_+$ and an attainable mixed-consumption plan bundle (strategy bundle)

$\{\mu^{*j}\}_{j\in N} \in F_{ic}$, such that each mixed commodity bundle μ^{*j} satisfies the *ex ante* expected budget constraint,

$$\sum_{t\in T} \pi(t) \int_{C^j} \sum_{h=1}^{l} p_h c_h^j \mu^{*j}[t](dc^j) \leq \sum_{h=1}^{l} p_h e_h^j,$$

and is the best of those mixed commodity bundles satisfying the *ex ante* expected budget constraint, that is, for any Bayesian incentive compatible mixed commodity bundle μ^j for which $\sum_{t\in T} \pi(t) \int_{C^j} \sum_{h=1}^{l} p_h c_h^j \mu^j[t](dc^j) \leq \sum_{h=1}^{l} p_h e_h^j$,

$$EU^j(\mu^j) \leq EU^j(\mu^{*j}).$$

Applying the standard argument, Forges, Heifetz and Minelli established that *there exists an ex ante competitive equilibrium, and each ex ante competitive allocation is an ex ante core allocation.*

Then they considered the replica economies *à la* Debreu and Scarf (1963), in which the type profile space in the q-fold replicated economy is the q-fold product of space T and the *ex ante* probability is the q-fold product probability of π. They pointed out by an example that due to the dependence of utility function u^j on the type profile t, *the equal-treatment property is not valid for an ex ante core allocation of the q-fold replicated economy*. However, they established that *in the case of no externalities ($u^j = u^j(c^j, t^j)$), the q-fold replicated economy has an ex ante core allocation with the equal treatment property, and a consumption plan bundle with the equal treatment property which is in the ex ante core of all q-fold replicated economies, $q = 1, 2, \cdots$, is an ex ante competitive allocation.*

We have already pointed out the weakness of the attainability condition in their definition of *ex ante* core allocation. More serious problems show up in their mediator-based approach to the *ex ante* competitive allocation: each mixed commodity bundle μ^j depends fully on T, so it is not clear how a consumer can choose his mixed demand contingent upon the others' private information while being uncommunicative, and the *ex ante* notions of attainability and budget constraint fail to guarantee the attainability and the budget constraint at the *interim* time of actually executing these consumption plans. More importantly, they have not provided a rationale for imposing Bayesian incentive compatibility on the competitive allocations. A competitive equilibrium is an outcome of a specific noncooperative behavior guided only by a price vector established in the market, each consumer chooses his mixed commodity bundle by himself without coordinating with other consumers, so there is no need for him to promise truthful execution of his strategy to anybody. Thus, the sensible setup would be that an *ex ante* core allocation satisfies both the private measurability condition (if we really want to avoid the mediator) and the Bayesian incentive compatibility condition 2.4.1, and an *ex ante* competitive allocation satisfies only the private measurability condition. It is not clear if Forges, Heifetz and Minelli's results still remain to be true when the two conditions are discriminatorily applied as suggested here. Finally, we repeat our position that a mixed-choice appoach avoids the question of explaining deterministic choice.

While Forges, Heifetz and Minelli (2001) studied an *ex ante* Bayesian incentive-compatible core allocation which may not be private measurable, Einy, Moreno and Shitoviz (2001a) studied an *ex ante* private measurable core allocation which may not be Bayesian incentive-compatible (a private core allocation of definition 2.5.5), in the Bayesian pure exchange economy with a nonatmic measure space of consumers $(A, \mathcal{A}, \nu)$ and a general finite state space $(\Omega, \mathcal{T}, \pi)$, $\#\Omega < \infty$, in which each consumer a has a private information structure as a subalgebra $\mathcal{T}^a$ of $\mathcal{T}$.

The state ω-contingent claim for commodity h is a commodity traded in the *ex ante* period which promises delivery of a unit of commodity h upon realization of state ω in the *interim* period, and no delivery upon realization of any other state. A claim allocation in A is a function $x : A \times \Omega \to \mathbf{R}^l_+$, assigning to each consumer a the claim bundle $x(a, \cdot)$, which is attainable in the economy,

$$\forall\ \omega \in \Omega : \int_A x(a, \omega)\nu(da) \leq \int_A e(a, \omega)\nu(da).$$

It is said to satisfy the private measurability condition if function $x(a, \cdot)$ is $\mathcal{T}^a$-measurable, ν-a.e. (see condition 2.3.1 for the null communication system). Einy, Moreno and Shitoviz considered Radner's (1968) *ex ante competitive equilibrium* of the state-contingent claim market defined as a pair (p^*, x^*) of price vector $p^* : \Omega \to \mathbf{R}^l_+$ and measurable claim allocation $x^* : A \times \Omega \to \mathbf{R}^l_+$ such that ν-a.e., consumer a's claim bundle $x^*(a, \cdot)$ maximizes his *ex ante* expected utility subject to the budget constraint:

$$\text{Maximize} \qquad \sum_{\omega \in \Omega} \pi^a(\omega) u(a, x(a, \omega), \omega)$$

$$\begin{array}{ll}\text{subject to} & x(a,\cdot) \text{ is } \mathcal{T}^a\text{-measurable,} \\ & \displaystyle\sum_{\omega\in\Omega} p^*(\omega)\cdot x(a,\omega) \le \sum_{\omega\in\Omega} p^*(\omega)\cdot e(a,\omega),\end{array}$$

and Walras' law is satisfied with equality,

$$\sum_{\omega\in\Omega} p^*(\omega)\cdot \int_A x^*(a,\omega)\nu(da) = \sum_{\omega\in\Omega} p^*(\omega)\cdot \int_A e(a,\omega)\nu(da).$$

Applying the standard argument, Einy, Moreno and Shitoviz established that *an ex ante competitive equilibrium of the state-contingent market exists, and the set of ex ante competitive allocations is identical to the set of private information core allocations*.

We will present Serrano, Vohra and Volij's (forthcoming) negative result. They considered Wilson's (1978) coarse core allocations (definition 2.5.1) in the Bayesian pure exchange economies replicated as in Debreu and Scarf (1963); they did not impose private measurability (condition 2.3.1 for the null communication system) or Bayesian incentive compatibility (condition 2.4.1 or 2.4.5).

As a competitive equilibrium concept for $\mathcal{E}_{pe}$ which belongs to the coarse core, they defined a constrained market equilibrium as follows. Let $\Delta := \{p : T \to \mathbf{R}^l_+ \mid \sum_{t\in T}\sum_{h=1}^{l} p_h(t) = 1\}$ be the price domain. Consumer j's consumption plan is a plan $x^j : T \to \mathbf{R}^l_+$. A consumption plan x^j gives rise to the *interim* expected utility given type $\bar{t}^j$, $Eu^j(x^j \mid \bar{t}^j) := \sum_{t\in T} u^j(x^j(t),t)\pi(t \mid \bar{t}^j)$. An allocation is a consumption plan bundle x such that the total demand is equal to the total supply at every type profile,

$$\forall\ t \in T:\ \sum_{j\in N} x^j(t) = \sum_{j\in N} e^j(t).$$

A *constrained market equilibrium* is a pair (p^*, x^*) of price vector p^* and allocation x^* such that for each consumer j and each information $\bar{t}^j$, x^{*j} maximizes his conditional expected utility given $\bar{t}^j$ subject to the budget constraint given $\bar{t}^j$:

$$\begin{array}{ll}\text{Maximize} & Eu^j(x^j \mid \bar{t}^j) \\ \text{subject to} & \displaystyle\sum_{t^{N\setminus\{j\}}} p(\bar{t}^j, t^{N\setminus\{j\}})\cdot x^j(\bar{t}^j, t^{N\setminus\{j\}}) \\ & \displaystyle\le \sum_{t^{N\setminus\{j\}}} p(\bar{t}^j, t^{N\setminus\{j\}})\cdot e^j(\bar{t}^j, t^{N\setminus\{j\}}).\end{array}$$

Apart from failing to satisfy the measurability requirement, it differs from Radner's (1968) *ex ante* competitive equilibrium of the state-contingent claim market in that it accommodates $\#T^j$ constrained maximization problems that each consumer j possibly faces at the *interim* period. It raises the following serious conceptual questions: If it is intended to be an *interim* equilibrium concept, player j, acting alone in the market in which everybody is anonymous, and knowing that his true type is $\bar{t}^j$, does not bother acting rationally in the unrealized event $E := \{t \in T \mid t^j \neq \bar{t}^j\}$, yet his actions in E influence the competitive equilibrium price vector p^*. If it is intended to be an *ex ante* equilibrium concept, player j, not knowing his true type yet, tries to maximize the sole objective function (the *ex ante* expected utility), but there is no reason why he should segment the market into $\#T^j$ submarkets. This point is all the more problematic since the way to segment the market differs among different anonymous consumers (commodity $(h, t^i, t^j, t^{N\setminus\{i,j\}})$ is traded with commodity $(k, t'^i, t^j, t^{N\setminus\{j\}})$ in consumer j's mind, yet they cannot be traded in consumer i's mind). We are, therefore, led back to Radner's *ex ante* competitive equilibrium of the state-contingent claim market as an appropriate *ex ante* equilibrium concept.

It is easy to check that *a constrained market equilibrium allocation is a coarse core allocation.*

Serrano, Vohra and Volij then constructed an example to assert that the Debreu-Scarf type core convergence theorem is not true for the coarse core: *The coarse core in a replica of size 2 of this example contains an allocation which does not satisfy the equal treatment property. There exists an allocation in this example whose m-replica is in the coarse core of the m-replicated economy for all m, yet it cannot be a constrained market equilibrium allocation.*

Einy, Moreno and Shitovitz (2001b) looked at two notions of the bargaining set of a Bayesian pure exchange economy with a nonatomic measure space of consumers, and established an equivalence result and a non-equivalence result, respectively, with respect to Radner's *ex ante* competitive allocations of the state-contingent market.

3.6 Other viewpoints

So far we have looked at analyses of situations where several players form a coalition within which to communicate each other and coordinate their strategy choice. Each member of a coalition knows the membership of his coalition, so he knows whom to deal with. In this subsection, however, we will briefly present two models that are based on another view on coalition formation; specifically they describe situations in which coalitional membership is anonymous.

Let $(A, \mathcal{A}, \nu)$ be a probability space of players. Demange and Guesnerie (2001) postulated that there exists a finite state space Ω; each player a's type is described by a member $\bar{\omega}(a)$ of Ω. The profile of the economy, $\bar{\omega} : A \to \Omega$, is not known, but its distribution $\bar{\pi} := \nu \circ (\bar{\omega})^{-1}$ is public knowledge as the *ex ante* objective probability on the types, and is identified with the grand coalition A since the players are anonymous. Let $\bar{\omega}|_S : S \to \Omega$ be the restriction of $\bar{\omega}$ to coalition $S \in \mathcal{A}$. Associated with each coalition S is the measure on Ω, $\bar{\pi}^S := \nu \circ (\bar{\omega}|_S)^{-1}$, representing its size $\bar{\pi}^S(\Omega)$ and distribution of types $\bar{\pi}^S(\cdot)$. Define, therefore, the space of type distributions,

$$\Pi := \{\pi : 2^\Omega \to \mathbf{R}_+ \mid \pi \text{ is additive, } \pi(\Omega) \leq 1\}.$$

The *interim* stage is defined as the period in which each player a knows his type $\bar{\omega}(a)$ as his private information, as well as the public information $\bar{\pi}$.

Denote by C the outcome space. The preference relation of the players of type ω is represented by a von Neumann-Morgenstern utility function, $u(\cdot, \omega) : C \to \mathbf{R}$.

Demange and Guesnerie's main concern was a mechanism design for the grand coalition. A *plan* associates with each type ω an outcome realized for ω. Denote by F^π the set of all feasible plans for type distributions π, *a priori* given to the model. A *mechanism* is a function $f : \Omega \times \Pi \to C$. There are several scenarios for how a mechanism works; the following is a typical one. An underwriter of a mechanism (e.g., the government, or an insurer, or a mediator) designs mechanism f and announces it in public. He then receives players' confidential responses about their private types, and forms the distribution of the reported types π. The player who reported type ω will then receive outcome $f(\omega, \pi)$. The Demange-Guesnerie mechanism can be called an *anonymous* mechanism, in that the outcome for this player is not determined by the exact identities of all respondents or the exact reported type of each respondent, but merely by the reported-type distribution in addition to his own report.

The grand coalition has the observable size $\pi(\Omega) = 1$, so define $\Pi(1) :=$

$\{\pi \in \Pi \mid \pi(\Omega) = 1\}$. Mechanism f is *feasible for A*, if

$$\forall\ \pi \in \Pi(1) : f(\cdot, \pi) \in F^{\pi}.$$

It is called *incentive-compatible for A*, if

$$\forall\ \omega, \omega' \in \Omega : \forall\ \pi \in \Pi(1) : u(f(\omega, \pi), \omega) \geq u(f(\omega', \pi), \omega).$$

Demange and Guesnerie proposed several coalitional stability concepts for an incentive-compatible, feasible mechanism f for A, and defined the associated core concepts. The basic scenario for the blocking behavior goes as follows: An underwriter for a blocking coalition S announces in public a new mechanism, and those players who are made better off with the new mechanism (compared with the standing mechanism f for A) respond by confidentially reporting his type to the underwriter, and form the blocking coalition S.

Given the type-distribution π of the respondents, the size $s = \pi(\Omega) \in [0, 1]$ of the respondents is observable to the underwriter, so define

$$\Pi(s) := \{\pi \mid \pi(\Omega) = s\}.$$

Also observable is the support $\operatorname{supp} \pi \subset \Omega$ of the reported type-distribution.

One stability concept assumes the situation in which each member of a blocking coalition S exactly knows its distribution $\bar{\pi}^S$. Let s be the size of S, $s := \bar{\pi}^S(\Omega)$. Coalition S *statistically blocks* mechanism f for the grand coalition, if there is a mechanism g that is feasible,

$$\forall\ \pi \in \Pi(s) : g(\cdot, \pi) \in F^{\pi},$$

and is incentive-compatible,

$$\forall\ \omega, \omega' \in \Omega : \forall\ \pi \in \Pi(s) : u(g(\omega, \pi), \omega) \geq u(g(\omega', \pi), \omega),$$

such that it improves upon f,

$$\forall\ \omega \in\ \operatorname{supp} \bar{\pi}^S : u(g(\omega, \bar{\pi}^S), \omega) > u(f(\omega, \bar{\pi}), \omega).$$

The *statistical core* is the set of incentive-compatible, feasible mechanisms for A that are not statistically blocked.

Another stability concept addresses the situation in which the type-distribution of a blocking coalition is not known, but the blocking mechanism provides self-selection criterion to reveal it. Formulation of this self-selection criterion is the original conceptual contribution that Demange and Guesnerie's present paper and Hara's paper (to be introduced in the latter half of this subsection) have made to the core analysis.

Notice that $u^*(\omega) := u(f(\omega, \bar{\pi}), \omega)$ is the status quo reservation level, given a mechanism f for the grand coalition A. The standing mechanism f is *u^*-beliefs blocked*, if there exists $\Omega' \subset \Omega$ and an incentive-compatible and feasible mechanism g for coalition $S := (\bar{\omega})^{-1}(\Omega')$ (the set of all players whose type is in Ω') such that

$$\begin{aligned} \forall\ \omega \in \Omega' &\ :\ u(g(\omega, \bar{\pi}^S), \omega) > u(f(\omega, \bar{\pi}), \omega), \text{ and} \\ \forall\ \omega \notin \Omega' &\ :\ u(g(\omega, \bar{\pi}^S), \omega) \leq u^*(\omega). \end{aligned}$$

The first set of inequalities says that the players in the blocking coalition S improve upon the standing outcomes. The second set of weak inequalities is the self-selection criterion: nobody outside coalition S has the incentive to join the blocking coalition. By announcing this mechanism g, the underwriter can form exactly the blocking coalition S. The *u^*-beliefs-based core* is the set of incentive-compatible, feasible mechanisms for A that are not u^*-beliefs blocked.

Hara (2001) recently proposed a new core concept for the static pure exchange economy $\mathcal{E}_{pe} := \{\mathbf{R}^l_+, u(a, \cdot), e(a)\}_{a \in A}$ with a nonatomic probability space of consumers $(A, \mathcal{A}, \nu)$; here the type profile space is a singleton, so notation for a type will be suppressed, and consumer a's preference relation is represented by a utility function of his consumption, $u(a, \cdot) : \mathbf{R}^l_+ \to \mathbf{R}$. Hara motivates his new core concept with the imaginary environment in which consumers have gathered in a marketplace, being aware of the statistical distribution of the others' characteristics, so each consumer knows that somewhere in the marketplace there is another consumer he can engage with in a mutually beneficial exchange but he cannot locate such a trading partner. In this environment, he can perhaps post a notice for the entire crowd of consumers to solicit such and such a unit of good A in exchange for such and such a unit of another good B. A coalition can then be formed with whoever comes forward to enter into this trade. (This scenario is also applicable to Demange and Guesnerie's (2001) model.)

For each coalition S, denote by F^S the set of all attainable allocations,

$$F^S := \left\{ f : S \to \mathbf{R}^l_+ \;\middle|\; \begin{array}{l} f \text{ is } \nu\text{-integrable}, \\ \int_S f d\nu = \int_S e d\nu \end{array} \right\}.$$

Allocation $f \in F^S$ is called *envy-free*, if there exists $S' \in \mathcal{A}$, $S' \subset S$ and $\nu(S') = \nu(S)$, such that

$$\begin{array}{l} (\forall a, b \in S' : e(a) + (f(b) - e(b)) \in \mathbf{R}^l_+) : \\ \qquad u(a, f(a)) \geq u(a, e(a) + (f(b) - e(b))). \end{array}$$

Let $f \in F^A$ be a standing envy-free allocation in the grand coalition. Each consumer a's status quo reservation level is then given as $u^*(a) := u(a, f(a))$. The standing allocation f is *blocked*, if there exist a coalition S with a positive measure ($S \in \mathcal{A}$, $\nu(S) > 0$) and its envy-free allocation $g \in F^S$, such that coalition S improves upon f via g, that is, there exists $S' \in \mathcal{A}$, $S' \subset S$ and $\nu(S') > 0$, for which

$$\begin{array}{rcl} u(a, g(a)) & \geq & u(a, f(a)),\ \nu\text{-a.e. in } S, \\ u(a, g(a)) & > & u(a, f(a)),\ \nu\text{-a.e. in } S', \end{array}$$

and such that allocation g satisfies the self-selection criterion *vis-à-vis* f in that no set of outsiders to S with a positive measure want to pretend that they were members of S, that is,

$$\begin{array}{l} \exists\, S' \in \mathcal{A} : S' \subset S,\ \nu(S') = \nu(S), \\ \exists\, T' \in \mathcal{A} : T' \subset A \setminus S,\ \nu(T') = \nu(A \setminus S), \\ (\forall a \in T' : \forall b \in S' : e(a) + (g(b) - e(b)) \in \mathbf{R}^l_+) : \\ \qquad u(a, e(a) + (g(b) - e(b))) \leq u^*(a). \end{array}$$

The *anonymous core* is the set of all envy-free allocations $f \in F^A$ that are not blocked.

Hara (2001) established an equivalence theorem between the anonymous core and the set of competitive allocations, and also a generic limit theorem for anonymous core for the replica finite economies.

Bibliography

Demange, G., and R. Guesnerie (2001): "On coalitional stability of anonymous interim mechanisms," *Economic Theory* **18**, 367-389.

Einy, E., D. Moreno, and B. Shitovitz (2000a): "On the core of an economy with differential information," *Journal of Economic Theory* **94**, 262-270.

Einy, E., D. Moreno, and B. Shitovitz (2000b): "Rational expectations equilibria and the ex-post core of an economy with asymmetric information," *Journal of Mathematical Economics* **34**, 527-535.

Einy, E., D. Moreno, and B. Shitovitz (2001a): "Competitive and core allocations in large economies with differential information," *Economic Theory* **18**, 321-332.

Einy, E., D. Moreno, and B. Shitovitz (2001b): "The bargaining set of a large economy with differential information," *Economic Theory* **18**, 473-484.

Forges, F. (year?): "Le cœur *ex ante* incitatif d'une économie d'échange en information asymétrique," *Revue économique*

Forges, F., A. Heifetz, and E. Minelli (2001): "Incentive compatible core and competitive equilibria in differential information economies," *Economic Theory* **18**, 349-365.

Forges, F., J.-F. Mertens, and Rajiv Vohra (2000): "The ex ante incentive compatible core in the absence of wealth effects," unpublished paper.

Forges, F., and E. Minelli (2001): "A note on the incentive compatible core," *Journal of Economic Theory* **98**, 179-188.

Forges, F., E. Minelli, and Rajiv Vohra (2000): "Incentives and the core of an exchange economy: A survey," unpublished paper.

Glycopantis, D., A. Muir, and N. C. Yannelis (2001): "Am extensive form interpretation of the private core," *Economic Theory* **18**, 293-319.

Hahn, G., and N. C. Yannelis (1997): "Efficiency and incentive compatibility in differential information economies," *Economic theory* **10**, 383-411.

Hahn, G., and N. C. Yannelis (2001): "Coalitional Bayesian Nash implementation in differential information economies," *Economic Theory* **18**, 485-509.

Hara, C. (2001): "The anonymous core of an exchange economy," unpublished paper, Faculty of Economics and Politics, University of Cambridge.

Ichiishi T. (1995): "Cooperative processing of information," in: T. Maruyama and W. Takahashi (eds.), *Nonlinear and Convex Analysis in Economic Theory*, Lecture Notes in Economics and Mathematical Systems, No. 419, pp. 101-117. Heidelberg/Berlin: Springer-Verlag.

Ichiishi T., and A. Idzik (1996): "Bayesian cooperative choice of strategies," *International Journal of Game Theory* **25**, 455-473

Ichiishi, T., A. Idzik, and J. Zhao (1994): "Cooperative processing of information via choice at an information set," *International Journal of Game Theory* **23**, 145-165.

Ichiishi, T., and S. Koray (2000): "Job matching: A multi-principal, multi-agent model," *Advances in Mathematical Economics* **2**, 41-66.

Ichiishi, T., and R. Radner (1999): "A profit-center game with incomplete information," *Review of Economic Design* **4**, 307-343.

Ichiishi, T., and M. Sertel (1998): "Cooperative *interim* contract and recontract: Chandler's M-form firm," *Economic Theory* **11**, 523-543.

Koutsougeras, L. C., and N. C. Yannelis (1993): "Incentive compatibility and information superiority of the core of an economy with differential information," *Economic Theory* **3**, 195-216.

Koutsougeras, L. C., and N. C. Yannelis (1999): "Bounded rational learning in differential information economies: Core and value," *Journal of Mathemtical Economics* **31**, 373-391.

Krasa, S.(1999): "Unimprovable allocations in economies with incomplete information," *Journal of Economic Theory* **87**, 144-168.

Krasa, S., and W. Shafer (2001): "Core concepts in economies where information is almost complete," *Economic Theory* **18**, 451-471.

Lee, D., and O. Volij (1996): "Rationality and the core of an economy with differential information," unpublished paper.

Lefebvre, I. (2001): "An alternative proof of the nonemptiness of the private core," *Economic Theory* **18**, 275-291.

Page, Jr., F. H. (1997): "Market games with differential information and infinite dimensional commodity spaces: The core," *Economic Theory* **9**, 151-159.

Page, Jr., F. H., and M. H. Wooders (1994): "Asymmetric information, the efficient core, and farsightedly stable trading mechanisms," unpublished paper.

Serfes, K. (2001): "Non-myopic learning in differential information economies: the core," *Economic Theory* **18**, 333-348.

Serrano, R., and Rajiv Vohra (2001): "Some limitations of virtual Bayesian implementation," *Econometrica* **69**, 785-792.

Serrano, R., Rajiv Vohra, and O. Volij (forthcoming): "On the failure of core convergence in economies with asymmetric information," *Econometrica*.

Vohra, Rajiv (1999): "Incomplete information, incentive compatibility, and the core," *Journal of Economic Theory* **86**, 123-147.

Volij, O. (2000): "Communication, credible improvements and the core of an economy with asymmetric information," *International Journal of Game Theory* **29**, 63-79.

Wilson, R. (1978): "Information efficiency, and the core of an economy," *Econometrica* **46**, 807-816.

Yamazaki, A., and T. Ichiishi (in preparation): "Incentive compatibility and the core of a large economy with differential information."

Yannelis, N. C. (1991): "The core of an economy with differential information," *Economic Theory* **1**, 183-198.

Yazar, J. (2001): "Ex ante contracting with endogenously determined communication plans," *Economic Theory* **18**, 439-450.

Related Works

Abreu, D., and H. Matsushima (1992): "Virtual implementation in iteratively undominated strategies: Complete information," *Econometrica* **60**, 993-1008.

Akerlof, G. (1970): "The market for lemons: Quality uncertainty and the market mechanisms," *Quarterly Journal of Economics* **84**, 488-500.

Aumann, R. J. (1964): "Markets with a continuum of traders," *Econometrica* **32**, 39-50.

Aumann, R. J. and B. Peleg (1960): "Von Neumann-Morgenstern solutions to cooperative games without side payments," *Bulletin of the American Mathematical Society* **66**, 173-179.

Chandler, Jr., A. D. (1962): *Strategy and Structure*, Cambridge, MA: MIT Press.

d'Aspremont, C., and L.-A. Gérard-Varet (1979): "Incentives and incomplete information," *Journal of Public Economics* **11**, 22-45.

Debreu, G., and H. Scarf (1963): "A limit theorem on the core of an economy," *International Economic Review* **4**, 235-246.

Harsanyi, J. C. (1967/1968): "Games with incomplete information played by 'Bayesian' players," *Management Science: Theory* **14**, 159-182 (Part I), 320-334 (Part II), 486-502 (Part III).

Ichiishi, T. (1993a): *The Cooperative Nature of the Firm*, Cambridge, U.K.: Cambridge University Press.

Ichiishi, T. (1993b): "The cooperative nature of the firm: Narrative," *Managerial and Decision Economics* **14** (1993), 383-407. In: Special Issue edited by Koji Okuguchi on Labor-Managed Firms Under Imperfect Competition (and Related Problems).

Koray, S., and M. R. Sertel (1992): "The welfarisitc characterization of two-person revelation equilibrium under imputational government," *Social Choice and Welfare*, **9**, 49-56.

Holmström, B., and R. B. Myerson (1983) "Efficient and durable decision rules with incomplete information," *Econometrica* **41**, 1799-1819.

Myerson, R. B. (1984): "Cooperative games with incomplete information," *International Journal of Game Theory* **13**, 69-96

Radner, R. (1968): "Competitive equilibrium under uncertainty," *Econometrica* **36**, 31-58.

Radner, R. (1979): "Rational expectations equilibrium generic existence and the information revealed by prices," *Econometrica* **47**, 655-678.

Radner, R. (1992): "Transfer payments and the core of a profit-center game," in: P. Dasgupta, *et al.* (eds.), *Economic Analysis of Markets and Games (Essays in Honor of Frank Hahn)*, pp. 316-339. Cambridge, MA: MIT Press.

Rosenmüller, J. (1992): "Fee games: (N)TU-games with incomplete information," in: R. Selten (ed.) *Rational Interaction: Essays in Honor of John C. Harsanyi*, pp.53-81. Berlin: Springer-Verlag.

Scarf, H. (1973): *The Computation of Economic Equilibria*, New Haven, CT: Yale Univ. Press.

Scarf, H. (1986): "Notes on the core of a productive economy," in W. Hildenbrand, *et al.* (eds.), *Contributions to Mathematical Economics (in Honor of Gerard Debreu)*, pp. 401-429. Amsterdam/New York: North-Holland.

Appendix

This Appendix presents the definition of Bayesian incentive compatibility for the two-*interim*-period model of Bayesian society which satisfies the information-revelation process (subsection 3.2.1). It extends d'Aspremont and Gérard-Varet's (1979) original definition (condition 2.4.1) to the present context. It also incorporates Murat Sertel's idea, *pretend-but-perform principle*, later developed by Koray and Sertel (see, e.g., Koray and Sertel, 1992). According to this principle, players are allowed to *pretend* to have their chosen types, *but* must thereafter *perform* so as not to belie them.

A *pretension function* of player j is a function $\sigma:\ T^j \to T^j$, which says that when his true type is t^j, he acts (makes a choice) as though his type were $\sigma(t^j)$. Given any algebra $\mathcal{B}^j$ on T^j, denote by $\mathrm{endo}(T^j, \mathcal{B}^j)$ the set[14] of all functions that map each t^j into the minimal set of $\mathcal{B}^j$ that contains t^j. If every member of coalition S has information structure $\mathcal{B}^j$ about player j's type, then j can only choose a pretension function $\sigma \in \mathrm{endo}(T^j, \mathcal{B}^j)$.

In order to define the Bayesian incentive compatibility, one needs to clarify first which choice of player j is *legal* in the sense that the other members of the coalition S cannot catch j's false pretension about his true type. Suppose the members of S are deliberating on whether or not to sign a contract $x^S \in F'^S(\bar{x})$. At the beginning of the first *interim* period, player j's information structure is given as $\mathcal{T}^j$, and no other member has any part of this information (that is, if $i \in S \setminus \{j\}$, then $\mathcal{T}^i \cap \mathcal{T}^j = \{\emptyset, T\}$). So, player j is not caught in the first month no matter which choice he makes from $\{x_1^j(t^j) \mid t^j \in T^j\}$; that is, he can make choice according to any pretension function $\sigma \in \mathrm{endo}(T^j, \{\emptyset, T^j\})$, so that when player j's true type is t^j, he makes the choice $x_1^j(\sigma(t^j))$. By acting according to the function $x_1^j \circ \sigma$, player j having his true type t^j passes on to all the other members of S the information that event $A := (x_1^j)^{-1}(x_1^j \circ \sigma(t^j))$ has occurred. This information may be false, that is, t^j may not be a member of A, but the other members take it as j's testimony about himself and expect that j will act according to this information in the second *interim* period, that is, j will have to make a choice from $x_2^j(A)$ in the second month. Therefore, j's pretension function in the second month has to be of the form $\tau \circ \sigma$ for some $\tau \in \mathrm{endo}(T^j, \mathcal{A}(x_1^j))$.

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[14] Let $\mathcal{P}$ be the partition of T^j that consists of the minimal nonempty members of $\mathcal{B}^j$. Then, $\mathrm{endo}(T^j, \mathcal{B}^j) := \{\sigma:\ T^j \to T^j \mid \forall\ P \in \mathcal{P}:\ \sigma(P) \subset P\}$.

When j chooses such a pair of pretension functions, $\sigma \in \mathrm{endo}(T^j, \{\emptyset, T^j\})$ and $\tau \in \mathrm{endo}(T^j, \mathcal{A}(x_1^j))$, the other members $i \in S \setminus \{j\}$, acting honestly, would make choices $(x_1^i(t^i), x_2^i(\sigma(t^j), t^{S\setminus\{j\}}))$, because x_2^i is $\hat{\mathcal{T}}^i(x_1^S)$-measurable and $(\sigma(t^j), t^{S\setminus\{j\}}) \in ((x_1^j)^{-1}(x_1^j \circ \sigma(t^j)), (x_1^{S\setminus\{j\}})^{-1}(x_1^{S\setminus\{j\}}(t^{S\setminus\{j\}})))$.

The present concept of *Bayesian incentive compatibility* says that player j cannot benefit from any pair of pretension functions that are not caught. Thus, the set of Bayesian incentive-compatible feasible strategy set $\hat{F}^S(\bar{x})$ is defined by: $x^S \in \hat{F}^S(\bar{x})$, iff $x^S \in F'^S(\bar{x})$, and for all $j \in S$, all $\sigma \in \mathrm{endo}(T^j, \{\emptyset, T^j\})$, all $\tau \in \mathrm{endo}(T^j, \mathcal{A}(x_1^j))$ and all $t \in T$,

$$\begin{aligned} &Eu^j(x^S,\ \bar{x}^{N\setminus S} \mid \hat{\mathcal{T}}^j(x_1^S))(t) \\ &\geq\ Eu^j((x_1^j \circ \sigma,\ x_2^j \circ (\tau \circ \sigma,\ \mathrm{id})),\ (x_1^{S\setminus\{j\}},\ x_2^{S\setminus\{j\}} \circ (\sigma,\ \mathrm{id})),\ \bar{x}^{N\setminus S} \\ &\qquad \mid \hat{\mathcal{T}}^j(x_1^S))(t), \end{aligned}$$

where id is the identity map on $T^{S\setminus\{j\}}$.